Lecture 4

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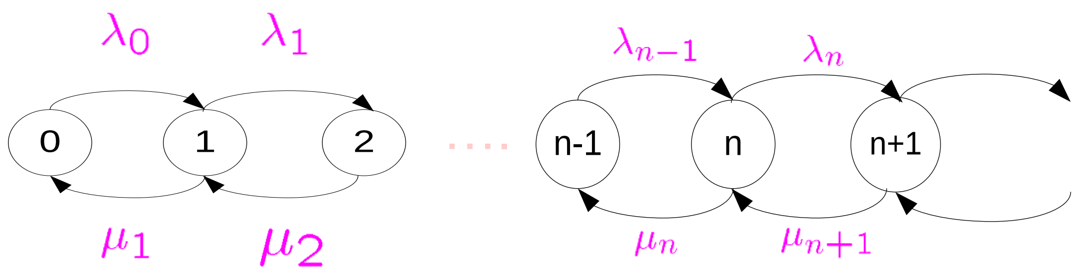
EE 336
Stochastic Models for the Analysis of Computer Systems
And Communication Networks

Outline

- Review of Lecture 3
- M/M/1/N queue
- M/M/m/m queue
- Engset Model
- M/M/1 with feedback
- Machine Repairman Model
- Some Application Examples

Review of Lecture 3 - 1

- Birth-death Processes:



$$\lambda_{n-1} p_{n-1} + \mu_{n+1} p_{n+1} = (\lambda_n + \mu_n) p_n$$ global balance Eqns.

$$\lambda_{n-1} p_{n-1} = \mu_n p_n$$ local (detailed) balance Eqns.

$$\Rightarrow \boxed{p_n = \frac{\lambda_{n-1}}{\mu_n} p_{n-1}}$$

get p_0 from $$\boxed{\left(1 + \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}}\right)^{-1} = p_0}$$

Review of Lecture 3 - 2

- M/M/1 queue:

$$\left.\begin{array}{ll} \lambda_n = \lambda & n \geq 0 \\ \mu_n = \mu & n \geq 1 \end{array}\right\}$$ M/M/1 queue with feedback is similar !!

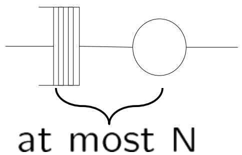
- M/M/1 queue (Infinite server queue):

$$\begin{array}{ll} \lambda_n = \lambda & n \geq 0 \\ \mu_n = n\mu & n \geq 1 \end{array}$$

- M/M/m queue:

$$\lambda_n = \lambda$$
$$\mu_n = min(n, m)\mu$$

- M/M/1/N queue: Finite buffer case



$$\begin{array}{ll} \lambda_n = \lambda & 0 \leq n \leq N-1 \\ \mu_n = \mu & 1 \leq n \leq N \end{array}$$

$$\boxed{p_n = \frac{\lambda}{\mu} p_{n-1} = \rho p_{n-1} = \rho^n p_0}$$

M/M/1/N Queue Analysis - 1

- M/M/1/N queue

$$\sum_{n=0}^{N} p_n = 1 \Rightarrow (1+\rho+\rho^2+\ldots+\rho^N)p_0 = 1$$

$$\boxed{\Rightarrow p_0 = \frac{1-\rho}{1-\rho^{N+1}}}$$ (note: as N$\to \infty, p_0 \to 1-\rho \ldots M/M/1 queue$)

$$\boxed{\therefore \quad p_n = \frac{(1-\rho)\rho^n}{1-\rho^{N+1}} \quad for \quad 0 \le n \le N}$$

p_N = prob. that the system is full

= prob. blocking $= p_B = \frac{(1-\rho)\rho^N}{1-\rho^{N+1}}$

Truncated modified geometric pmf

$$\boxed{G(z) = \sum_{n=0}^{N} \frac{\rho^n z^n}{1-\rho^{N+1}}(1-\rho) = \frac{(1-\rho)[1-(\rho z)^{N+1}]}{(1-\rho^{N+1})(1-\rho z)}}$$

M/M/1/N Queue Analysis - 2

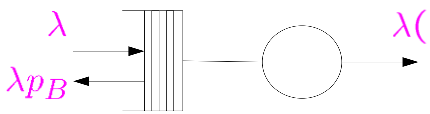
- Average queue length

$$\boxed{\begin{aligned} Q &= \frac{dG(z)}{dz}\Big|_{z=1} = \frac{(1-\rho)\,[-(N+1)\rho^{N+1}z^{N}(1-\rho z)+\rho(1-(\rho z)^{N+1})]}{(1-\rho^{N+1})\,(1-\rho z)^{2}}\Big|_{z=1} \\ &= \frac{\rho}{(1-\rho)} - \frac{(N+1)\rho^{N+1}}{(1-\rho^{N+1})} = \frac{\rho}{(1-\rho)}[1-(N+1)\overbrace{\frac{\rho^{N}(1-\rho)}{(1-\rho^{N+1})}}^{P_B}] \\ &= \frac{\rho}{(1-\rho)}[1-(N+1)P_B] = Q_{M/M/1}[1-(N+1)P_B] \rightarrow Q_{M/M/1} \ as\ N \rightarrow \infty \end{aligned}}$$

- Throughput

$$X = \sum_{n=1}^{N} \mu_n p_n = \mu \sum_{n=1}^{N} p_n = \mu(1-p_0)$$

$$= \mu[1-\frac{1-\rho}{1-\rho^{N+1}}] = \rho\mu(\frac{1-\rho^{N}}{1-\rho^{N+1}}) = \lambda[1-P_B]$$



$\lambda(1-p_B) = throughput = \mu(1-p_0)$

M/M/1/N Queue Analysis - 3

So, in M/M/1/N queue $\boxed{\rho(1-P_B) = 1 - p_0}$

$$P_B = P_N = \frac{(1-\rho)\rho^N}{1-\rho^{N+1}} \approx (1-\rho)\rho^N \;\; for\; N \gg 1 \,\&\, \rho < 1$$

$$\Rightarrow \rho^N = \frac{P_B}{1-\rho} \quad or \quad \boxed{N = \frac{ln(\frac{P_B}{1-\rho})}{ln\rho}}$$

Can use this relation to design buffer size.

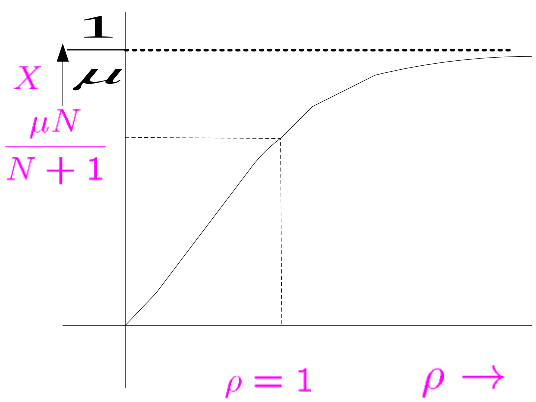
For example

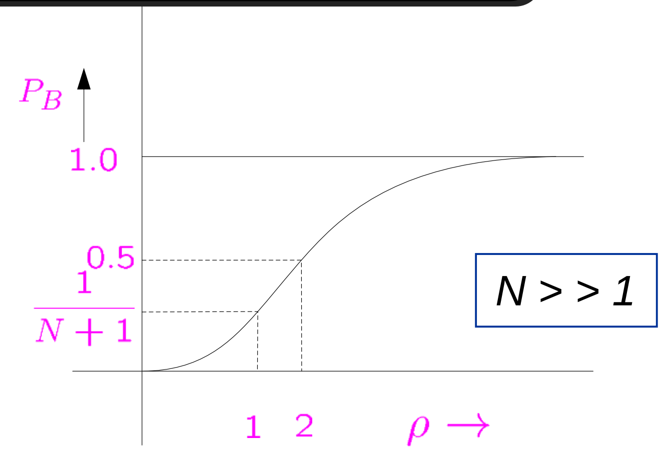
$$\rho = 0.5,\; P_B = 10^{-3} \Rightarrow N = \frac{-2.7}{-0.3} \approx 9$$

$$\rho = 0.5,\; P_B = 10^{-6} \Rightarrow N = \frac{-5.7}{-0.3} \approx 19$$

- Note that ρ can be >1. If ρ >1, change $\boxed{p_0 = \frac{\rho - 1}{\rho^{N+1} - 1}}$ and don't actually have a control over P_B since $\boxed{P_B = p_N = \frac{(\rho-1)\rho^N}{\rho^{N+1}-1} \approx \frac{\rho-1}{\rho} \; as \; \rho \to \infty}$

M/M/1/N Queue Analysis - 4





- Average response time:

$$R = \frac{Q}{X} = \frac{\left(\frac{\rho}{1-\rho}\right)(1-P_B) - \frac{N\rho}{1-\rho}P_B}{\lambda(1-P_B)}$$

$$\therefore \quad \boxed{R = \frac{1}{\mu}\frac{1}{1-\rho} - \frac{N}{\mu(1-\rho)}\frac{P_B}{1-P_B}}$$

(or)

$$\boxed{R_{M/M/1/N} = R_{M/M/1}\left[1 - \frac{NP_B}{1-P_B}\right] = \frac{R_{M/M/1}}{1-P_B}\left[1-(N+1)P_B\right]}$$

M/M/1/N Queue Analysis - 5

When $\rho >> 1,\; Q \to N - \frac{1}{\rho - 1} = N - \frac{1}{\rho P_B}$

$$R \to \frac{N}{\mu} - \frac{1}{\mu(\rho - 1)}$$

- Utilization:

$$U = \frac{X}{\mu} = \rho(1 - P_B)$$

- Average waiting time

$$W = R - \frac{1}{\mu} = \frac{1}{\mu}\frac{\rho}{1-\rho} - \frac{NP_B}{\mu(1-\rho)(1-P_B)}$$

- Average waiting queue length

$$Q_W = WX = \frac{\rho^2}{1-\rho}(1-P_B) - \frac{NP_B \cdot \rho}{(1-\rho)}$$

M/M/m/m Queue Analysis - 1

- M/M/m/m: the m-server loss system: valid for *M/G/m/m* system also

$$\lambda_n = \lambda, \ 0 \le n \le m-1$$

$$\mu_n = n\mu, \ 1 \le n \le m$$

$$p_n = \frac{\lambda}{n\mu} p_{n-1} \Rightarrow \ p_n = \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} p_0$$

So, $p_0 = \left[\sum_{n=0}^{m} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!}\right]^{-1}$

$$p_m = prob. \ \{all \ servers \ busy\}$$

$$= \frac{\left(\frac{\lambda}{\mu}\right)^m \frac{1}{m!}}{\sum_{n=0}^{m} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!}} = \frac{\frac{\rho^m}{m!}}{\sum_{n=0}^{m} \rho^n \frac{1}{n!}} = P_B$$

similar to truncated Poisson

This is known as Erlang's B-formula

- For $5 < \rho < 50$, useful approximations for m for a specified P_B are:

$$P_B = .01 \ \Rightarrow \ m \approx 5.5 + 1.17\rho$$

$$P_B = .001 \ \Rightarrow \ m \approx 7.8 + 1.28\rho$$

M/M/m/m Queue Analysis - 2

- Average queue length

$$Q=\sum_{n=1}^{m} np_n=\frac{\sum_{n=1}^{m}\frac{\rho^n}{(n-1)!}}{\sum_{n=0}^{m}\frac{\rho^n}{n!}}=\rho[1-P_m]=\rho[1-P_B]$$

$$=\textbf{Average number of busy servers}$$

Note that as $\rho\uparrow$ $P_B\uparrow$ & Q $\rightarrow m$ as it should!

- Throughput: $X=\lambda(1-P_B)=\lambda(1-P_m)$

Alternatively,

$$X=\sum_{n-1}^{m}\mu_n p_n=\frac{\mu\sum_{n=1}^{m}\frac{\rho^n}{(n-1)!}}{\sum_{n=0}^{m}\frac{\rho^n}{n!}}=\mu Q=\lambda(1-P_B)$$

- Average response time = $R=\frac{1}{\mu}$
- Average waiting time = $W = 0$
- Average waiting queue length = $Q_W = 0$
- Utilization $U = \frac{\lambda(1-P_B)}{m\mu}=\frac{\rho(1-P_B)}{m}=\frac{Q}{m}$

Engset Model Analysis - 1

- Engset Model

$$\lambda_n = (M-n)\lambda \quad 0 \le n \le N$$

$$\mu_n = n\mu \quad N \le M$$

$$\Rightarrow \quad p_n = (\frac{n-1}{n})\frac{\lambda}{\mu}p_{n-1} = (\frac{n-1}{n})\rho \cdot p_{n-1}$$

So, $p_n = \begin{pmatrix} M \\ n \end{pmatrix} \rho^n p_0 \quad where \quad \begin{pmatrix} M \\ n \end{pmatrix} = \dfrac{M!}{n!(M-n)!}$

$$p_0 = [\sum_{n=0}^{N} \begin{pmatrix} M \\ n \end{pmatrix} \rho^n]^{-1}$$

$$\boxed{p_n = \frac{\begin{pmatrix} M \\ n \end{pmatrix} \rho^n}{\left[\sum_{m=0}^{N} \begin{pmatrix} M \\ m \end{pmatrix} \rho^m\right]} \quad ; n = 0, 1, 2, \ldots, N}$$

called Engset distribution

Engset Model Analysis - 2

- For M>N, the blocking probability

$$P_B = P_N = \frac{\begin{pmatrix} M \\ N \end{pmatrix} \rho^N}{\left[\sum_{n=0}^{N} \begin{pmatrix} M \\ n \end{pmatrix} \rho^n\right]}$$

- Average queue length

$$Q = \sum_{n=1}^{N} n p_n$$

$p_0 = p_0(M)$ to emphasize its dependence on M for a given N

$$= p_0(M) \sum_{n=1}^{N} \frac{M!}{(n-1)!(M-n)!} \rho^n; \; p_0(M) = \left[\sum_{n=0}^{N} \begin{pmatrix} M \\ n \end{pmatrix} \rho^n\right]^{-1}$$

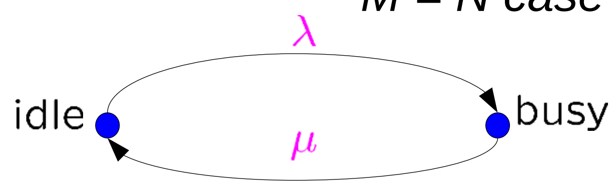
$$= M\rho \cdot p_0(M) \sum_{n=0}^{N-1} \frac{M-1!}{n!(M-1-n)!} \rho^n$$

$$= \frac{M\rho \cdot p_0(M)}{p_0(M-1)} [1 - P_B(M-1)] = \textbf{Average \# of busy lines}$$

Engset Model Analysis - 3

- Throughput $X = \sum_{n=1}^{N} n\mu p_n = \mu Q = \frac{M\lambda \cdot p_0(M)}{p_0(M-1)}[1 - P_B(M-1)]$
- Response time $R = \frac{1}{\mu}$
- Utilization $U = \frac{X}{N\mu} = \frac{Q}{N}$

M = N case



Note: 1) As *M* ! 1, & *λ* ! 0 3 *M λ* ! *λ'*, the measures reduce to those for *M/M/m/m* system: $Q = \frac{\lambda'}{\mu}(1 - P_B)$; $X = \lambda'(1 - P_B)$

2) *M=N*

$$\Rightarrow p_n = \frac{\binom{N}{n}\rho^n}{\left[\sum_{m=0}^{N}\binom{N}{m}\rho^m\right]} = \binom{N}{n} a^n(1-a)^{N-n};\ a = \frac{\rho}{1+\rho} = \frac{\lambda}{\lambda+\mu}$$

Binomial Distribution

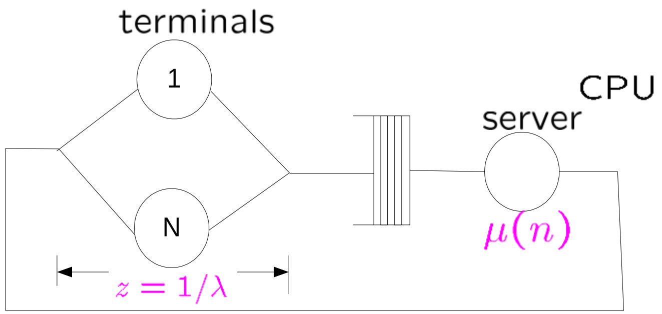
$\Rightarrow$ each server is either busy or idle & independent of others

$\Rightarrow$ performance measures

$$Q = Na = \frac{N\rho}{1+\rho};\ X = \frac{N\lambda}{1+\rho} = \mu Q;\ R = \frac{1}{\mu};\ U = \frac{Q}{N} = a = \frac{\rho}{1+\rho}$$

Analysis of Machine Repairman Model - 1

- Machine Repairman Model (simplest form of closed network)



When n user requests are at the CPU, (N-n) are *potential* user requests, so that

$$\lambda_n = \begin{cases} (N-n)\lambda & 0 \le n \le N-1 \\ 0 & for\ n \ge N \end{cases}$$

$$\mu(n) = \mu\ \{\textbf{constant}\}$$

showed earlier that:

$$\frac{N}{z+\frac{N}{\mu}} \le X(N) \le min\left\{\mu, \frac{N}{z+1/\mu}\right\}$$

$$max[N/\mu,\ z+1/\mu] \le R(N) \le z + N/\mu$$

In general: $p_n = \frac{(N-n+1)\lambda}{\mu(n)} p_{n-1};\quad n = 1, 2, ..., N$

For later notation, we write it as:

$$p(n/N) = \frac{(N-n+1)\lambda}{\mu(n)} p(n-1/N);\quad n = 1, 2, ..., N$$

$$= \frac{N!}{(N-n)!} \cdot \frac{\lambda^n}{\prod_{i=1}^{n} \mu(i)} p(o/N)$$

Analysis of Machine Repairman Model - 2

$$p(o/N) = \left[\sum_{n=0}^{N} \frac{N!}{(N-n)!} \cdot \frac{\lambda^n}{\prod_{i=1}^{n} \mu(i)}\right]^{-1}$$

Single server: $\mu(n) = \mu \Rightarrow p(o/N) = \left[\sum_{n=0}^{N} \frac{N!}{(N-n)!} \cdot \rho^n\right]^{-1} = [G(N)]^{-1}$

$$p(n/N) = \frac{N!}{(N-n)!} \cdot \rho^n \cdot p(o/N)$$

Infinite server: $\mu(n) = n\mu \Rightarrow p(o/N) = \left[\sum_{n=0}^{N} \frac{N!}{(N-n)!n!}\rho^n\right]^{-1} = [1+\rho]^{-N}$

Binomial Distribution $p(n/N) = \binom{N}{n} a^n (1-a)^{N-n}; \quad a = \frac{\rho}{1+\rho}$

Engset Model
M = N case

Let us consider the single server case in detail.

- Recursive expression for $G(N)$

$$G(N) = \sum_{n=0}^{N} \frac{N!}{(N-n)!}\rho^n = 1 + N\rho + N(N-1)\rho^2 + ... + N!\rho^N$$

$$= 1 + N\rho[1 + (N-1)\rho + ... + (N-1)!\rho^{N-1}]$$

$$= 1 + N\rho G(N-1); \quad G(0) = 1$$

Later, we will see that this kind of recursion extends to networks also and forms the basis of the so called “*Convolution Algorithm*”

Analysis of Machine Repairman Model - 3

- Throughput $X(N) = \sum_{n=1}^{N} \mu(n)p(n/N)$

$$= \mu(1 - p(o/N))$$

$$= \mu U(N)$$

- Utilization, *U(N)*= $\dfrac{\textbf{Throughput}}{\textbf{Max service rate}} = \dfrac{X(N)}{\mu}$

Also $X(N) = \mu\left[1 - \dfrac{1}{G(N)}\right] = \mu\cdot\dfrac{G(N)-1}{G(N)} = N\lambda\cdot\dfrac{G(N-1)}{G(N)} = \lambda\dfrac{\tilde{G}(N-1)}{\tilde{G}(N)}$

$$\boxed{U(N) = \frac{G(N)-1}{G(N)} = \rho\frac{\tilde{G}(N-1)}{\tilde{G}(N)}}$$

$$\boxed{\tilde{G}(N) = \frac{G(N)}{N!}}$$

- Mean value analysis (MVA) recursion:

Know $p(n/N) = \dfrac{N!}{(N-n)!}\rho^n \cdot \dfrac{1}{G(N)} = \dfrac{\rho^n}{(N-n)!}\cdot\dfrac{1}{\tilde{G}(N)}$

$$p(n-1/N-1) = \frac{(N-1)!}{(N-n)!}\rho^{n-1}\cdot\frac{1}{G(N-1)} = \frac{\rho^{n-1}}{(N-n)!}\cdot\frac{1}{\tilde{G}(N-1)}$$

$$\boxed{\frac{p(n/N)}{p(n-1/N-1)} = \frac{N\rho\cdot G(N-1)}{G(N)} = \frac{G(N)-1}{G(N)} = \rho\frac{\tilde{G}(N-1)}{\tilde{G}(N)} = U(N)}$$

Analysis of Machine Repairman Model - 4

)

$$\begin{aligned} p(n/N) &= U(N) \cdot p(n-1|N-1) \\ &= \frac{X(N)}{\mu} \cdot p(n-1|N-1) \end{aligned}$$

• In words, the probability distribution with *N* customers is related to the probability distribution of the same system with (*N-1*) customers. This observation is valid in a more general context involving multiple nodes and forms the basis of Reiser and Lavenberg's Mean Value Analysis (*MVA)* Algorithm.

Pursuing the recursion further,

- Average Queue length

$$\begin{aligned} Q(N) &= \sum_{n=1}^{N} np(n/N) \\ &= \frac{X(N)}{\mu} \sum_{n=1}^{N} np(n-1/N-1) \\ &= \frac{X(N)}{\mu} [\sum_{n=1}^{N} (n-1)p(n-1/N-1) + \sum_{n=1}^{N} p(n-1/N-1)] \\ &= \frac{X(N)}{\mu} [1 + Q(N-1)] \end{aligned}$$

Analysis of Machine Repairman Model - 5

Or

$$R_c(N) = \frac{1}{\mu}[1 + Q(N-1)]$$

valid for networks also

• In words, the mean response times of an arriving customer is equal to his own service time $1/\mu$ plus the waiting time due to customers in a system with (N-1) customers $\Rightarrow$ an arriving customer in a system with N terminals "sees" $Q(N$-$1)$ in equilibrium. We will see presently that *in an M/M/1 system (or so called open systems) an arriving and departing customers "see" the same # of customers (so called Burke's theorem). Not so in a closed system. In a closed system, an arrival sees a network with one less customer (himself removed)… so called "arrival theorem of closed systems".*

• Implementation (no closed form expression $\Rightarrow$ need to evaluate via a recursive algorithm)

Analysis of Machine Repairman Model - 6

$$
\left.
\begin{aligned}
&Q \leftarrow 0 \\
&\text{DO } n = 1, 2, \ldots, N \\
&\quad R_c \leftarrow \frac{1}{\mu}(1+Q) \\
&\quad X \leftarrow \frac{n}{R_c + \frac{1}{\lambda}} \\
&\quad Q \leftarrow X R_c \\
&\text{END DO} \\
&\quad U(N) = \frac{X}{\mu}
\end{aligned}
\right\} \Rightarrow
\begin{array}{c}
Q(N) \\
R_c(N) \\
X(N)
\end{array}
$$

- Questions:
  1. How to compute response time distributions? … we will do it only for M/M/1
  2. What does it all mean? … B-D processes are <u>"time-reversible Markov chains"</u>. We will pursue this in next class.

Response-time Density of M/M/1 Queue -1

- In discussing *M/M/1* queue, we made no assumption regarding queuing discipline

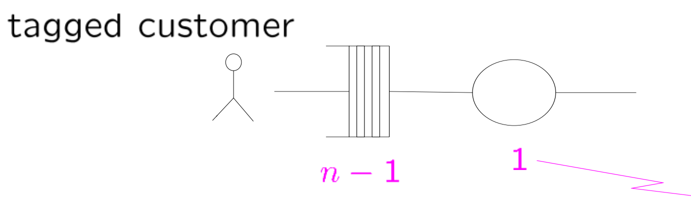
$\Rightarrow$ All queue disciplines yield identical *mean response times and mean queue lengths*

However, the distribution of response times does depend on the queue discipline.

We will compute the distributions for FCFS discipline.

Let us denote

$$F_R(r/n) \triangleq \mathrm{P}\{\text{the response time of tagged customer} \leq r \,|\, n \text{ customers are found in the system upon arrival}\}$$



$$R = \underbrace{t}_{\text{service time of tagged customer}} + \underbrace{t_1 + t_2 + \ldots + t_{n-1}}_{\text{service time of customers in queue}} + \underbrace{t_n'}_{\text{remaining (or residual) service time of customer in service}}$$

- Since service time is exponentially distributed t_n' has the same density (recall memory-less property)

Response-time Density of M/M/1 Queue -2

$\Rightarrow$ $t, t_1, t_2, \ldots, t_{n-1}, t_n^{'}$ are i.i.d. random variables

$$
\begin{aligned}
L_R(s/n) &= \left(\frac{\mu}{s+\mu}\right)^{n+1} \\
L_R(s) &= \sum_{n=0}^{\infty} L_R(s/n) p_n \\
&= \sum_{n=0}^{\infty} \left(\frac{\mu}{s+\mu}\right)^{n+1} (1-\rho)\rho^n \\
&= \left(\frac{\mu}{s+\mu}\right)(1-\rho)\frac{1}{1-\frac{\rho\mu}{s+\mu}} = \frac{\mu(1-\rho)}{[s+\mu(1-\rho)]}
\end{aligned}
$$

$\Rightarrow$ Response time is exponentially distributed with mean $\dfrac{1}{\mu(1-\rho)}$

$$f_R(t) = \mu(1-\rho)e^{-\mu(1-\rho)t}$$

Examples:

1). Suppose have M/M/1 queue and increase λ and μ by a factor k>1

$$\Rightarrow \quad \rho = \frac{k\lambda}{k\mu} \text{ same}$$

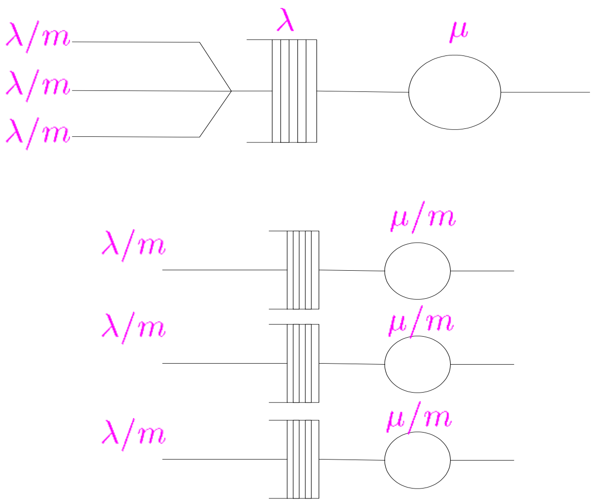
$$\Rightarrow Q = \frac{\rho}{1-\rho} \text{ same}$$

Application Examples -1

But $R = \dfrac{1}{k\mu(1-\rho)} \Rightarrow$ reduced by a factor of k.

An arriving customer sees the same # of customers in the steady state, but customers will be moving *k* times faster.

2). Statistical multiplexing vs. Time division or Frequency division multiplexing



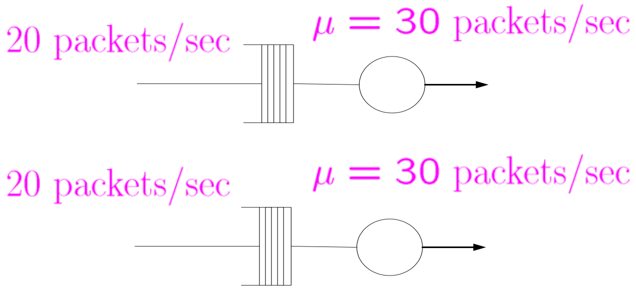
$$\Rightarrow R = \frac{1}{\mu(1-\rho)} = \frac{1}{\mu} + \frac{1}{\mu}\cdot\frac{\rho}{1-\rho}$$

$$\Rightarrow R = \frac{m}{\mu(1-\rho)} \Rightarrow \text{Large delay}$$
$$= \frac{m}{\mu} + \frac{m\rho}{1-\rho}\cdot\frac{1}{\mu}$$

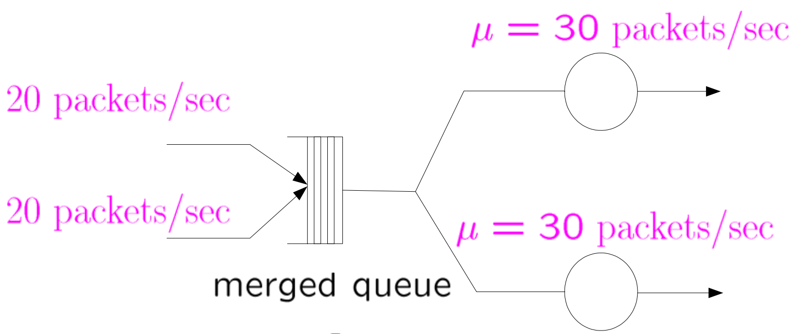
True only for Poisson streams
If arrivals are regular, time & frequency division multiplexing can be good

Application Examples - 2

3). Single queue versus separate queues (OR) why banks have single queues?



seperate $\Rightarrow R_{S_1} = R_{S_2} = \dfrac{1}{30(1-\frac{2}{3})} = .1$ sec



$$\rho = \frac{2}{3}$$

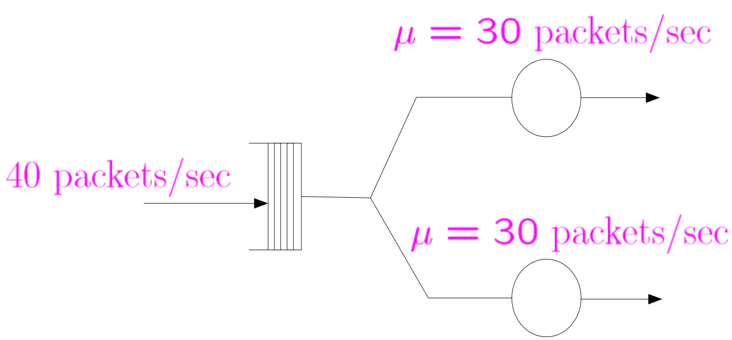
$$p_0 = \left(1 + \frac{4}{3} + \left(\frac{4}{3}\right)^2 \cdot \frac{1}{2} \cdot 3\right)^{-1} = .2$$

Application Examples - 3

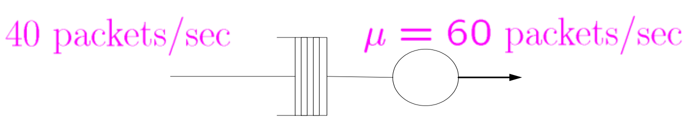
$$P_Q = .2 \cdot \frac{16}{9} \cdot \frac{1}{2} \cdot 3 = \frac{1.6}{3} = .533$$

$$R_C = \frac{1}{30} + \frac{.533}{30 \cdot \frac{1}{3}} = \frac{1}{30}(1 + 1.6) = .0867 < R_S \quad \text{(merged queue is better)}$$

4). Should I buy several small severs or one large one?



$$R_C = .0867$$



$$R_{LS} = \frac{1}{60 - 40} = .05 < R_c$$

- What happens to waiting time?

Application Examples - 4

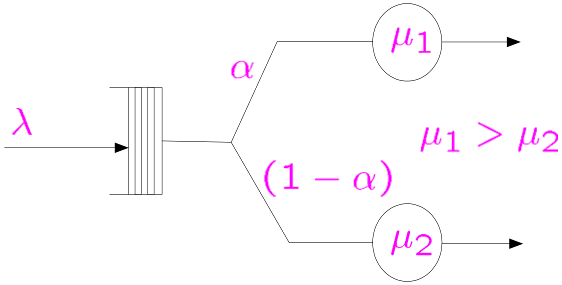
$$W_C = \frac{1}{60-40} \times .53 = .0265$$

$$W_{LS} = \frac{2/3}{60/3} = .0334 \quad \Rightarrow \quad W_{LS} > W_C \text{ but } R_{LS} < R_C$$

Buy the largest !!

But, from reliability viewpoint, several small ones may be better!!

5) Optimal Routing



$$R_1 = \frac{1}{\mu_1 - \lambda\alpha}; R_2 = \frac{1}{\mu_2 - \lambda(1-\alpha)}$$

$$R = \alpha R_1 + (1-\alpha) R_2 = \frac{\alpha}{\mu_1 - \lambda\alpha} + \frac{(1-\alpha)}{\mu_2 - \lambda(1-\alpha)}$$

$$Optimum \Rightarrow \frac{dR}{d\alpha} = 0 \Rightarrow \alpha^* = \frac{\sqrt{\frac{\mu_1}{\mu_2}} + \frac{\lambda}{\mu_2} - 1}{\frac{\lambda}{\sqrt{\mu_1\mu_2}} + \frac{\lambda}{\mu_2}}$$

$$\lambda = 4;\ \mu_1 = 4;\ \mu_2 = 1$$

$$\alpha^* = \frac{\sqrt{\frac{\mu_1}{\mu_2}} + \frac{\lambda}{\mu_2} - 1}{\frac{\lambda}{\sqrt{\mu_1\mu_2}} + \frac{\lambda}{\mu_2}} = \frac{5}{6}$$

Summary

- M/M/1/N queue
- M/M/m/m queue
- Engset Model
- M/M/1 with feedback
- Machine Repairman Model
- Some Application Examples