Lecture 1: Introduction & Mathematical Modeling

Prof. Krishna R. Pattipati
Prof. David L. Kleinman

Dept. of Electrical and Computer Engineering
University of Connecticut
Contact: krishna@engr.uconn.edu (860) 486-2890

ECE 6095/4121
Digital Control of Mechatronic Systems

Introduction

- Contact Information
  - Room number: ITE 350
  - Tel/Fax: (860) 486-2890/5585
  - E-mail: krishna@engr.uconn.edu

- Office Hours: Tuesday – Wednesday: 11:00-12:00 Noon

- **Very** demanding course
  - Homework every week and Design Projects (50%)
  - Sensors and Actuators Presentation (10%)
  - Class project (20%)
  - Paper presentation (5%)
  - Midterm – Take Home (15%)
- Course Materials: http://huskyct.uconn.edu

Background Needed

- Expected Background Knowledge (ECE 5101, ECE3111)
  - Differential equations
  - Continuous-time system modeling methods
    - Transfer functions, state-space models (canonical forms, minimal realizations)
    - Controllability & Observability
    - Transient response, especially for 2^{nd}-order systems
  - Stability theory for continuous-time systems
    - Feedback, Routh-Hurwitz, Lyapunov Theory
  - Graphical Tools
    - Bode plot, Nyquist plot, Nichols Chart, and Root-locus
  - Some basic knowledge of discrete-time systems
    - Z-transforms, difference equations, and signal sampling
  - Matrix theory and Linear Algebra
  - Knowledge and use of MATLAB

Mechatronic Systems Introduction & Overview

1. What is Mechatronics?
2. Elements of Mechatronics
3. Mechatronics Applications
4. Example of Mechatronics Systems
5. Mathematical Modeling of Mechatronic Systems
   - Diesel Engine Driving a Pump
   - Armature-controlled DC Motor
   - Magnetic Levitation
   - Inverted Pendulum
   - Induction Motor Spray Painting in an Automotive Plant
6. Different Mathematical Representations of Systems

What is Mechatronics? - 1

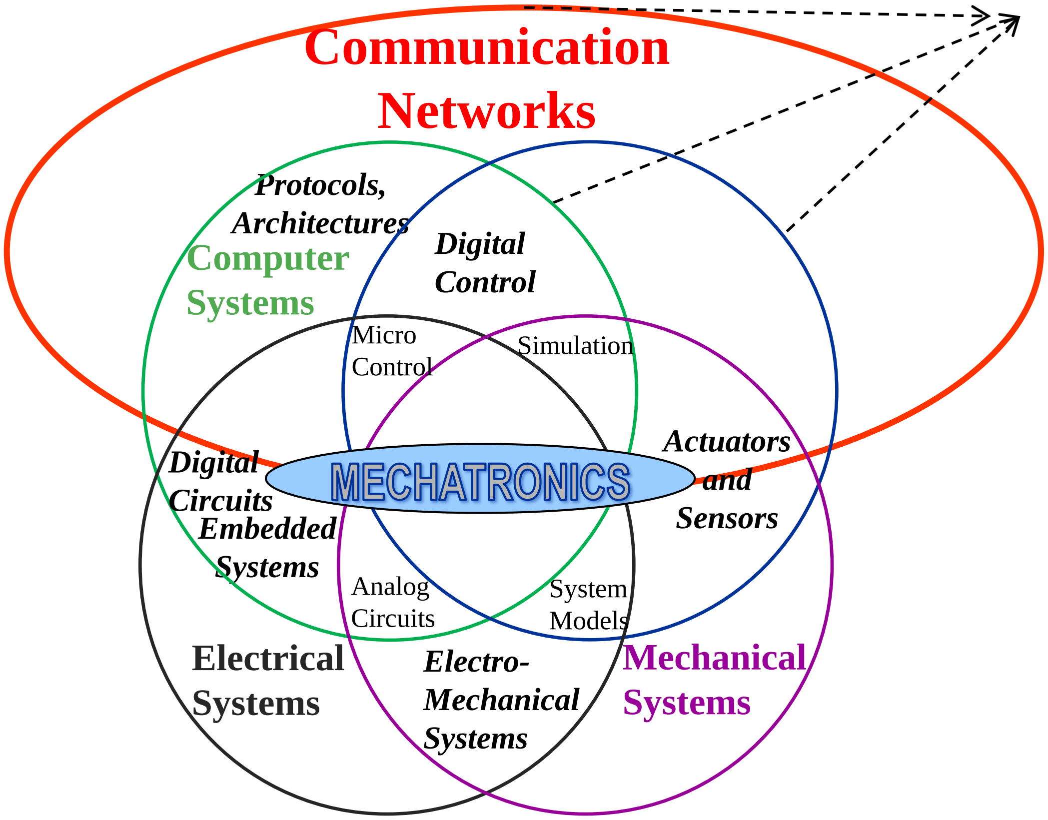
- The term **"Mechatronics"** was first coined by Tetsuro Mori, a senior engineer of the Japanese company Yasakawa*, in 1969
  - T. Mori, *"Mechatronics," Yasakawa Internal Trademark Application Memo, 21.131.01, July 12, 1969.*
  - R. Comerford, *"Mecha … what?" IEEE Spectrum, 31(8), 46-49, 1994.*
- **Mechatronics** refers to electro-mechanical systems and is centered on mechanics, electronics, computing and control which, when combined, make possible the generation of simpler, more economical, reliable and versatile systems
- **Mechatronics** is the synergistic integration of mechanical engineering, electronics and intelligent computer control in design and manufacture of products and processes
  - F. Harshama, M. Tomizuka, and T. Fukuda, *"Mechatronics-what is it, why, and how?-and editorial," IEEE/ASME Trans. on Mechatronics, 1(1), 1-4, 1996.*
- Synergistic use of precision engineering, control theory, computer science, and sensor and actuator technology to design improved products and processes."
  - S. Ashley, "*Getting a hold on mechatronics*," *Mechanical Engineering*, 119(5), 1997.

* Makes servos, machine controllers, AC motor drives, switches and robots

What is Mechatronics? - 2

- Field of study involving the analysis, design, synthesis, and selection of systems that combine electronics and mechanical components with modern controls and microprocessors.
  - D. G. Alciatore and M. B. Histand, *Introduction to Mechatronics and Measurement Systems, McGraw Hill, 1998.*
  - Good site for mechatronics: http://www.engr.colostate.edu/~dga/mechatronics/definitio*ns.html*
- Our working definition: Mechatronics is the synergistic integration of sensors, actuators, signal conditioning, power electronics, decision and control algorithms, and computer hardware and software to manage complexity, uncertainty, and communication in engineered systems.
- **An Embedded system**, a component of mechatronics system, is a combination of hardware and software designed to run on its own without human intervention, and may be required to respond to events in real-time
- When these systems are networked, they are called **Cyber-physical systems**
  - Numerous applications: Zero-accident highways, smart grid, smart buildings, tele-operation rooms, aerospace and transportation, robotics and intelligent machines,…..

A Venn Diagram View of Mechatronics



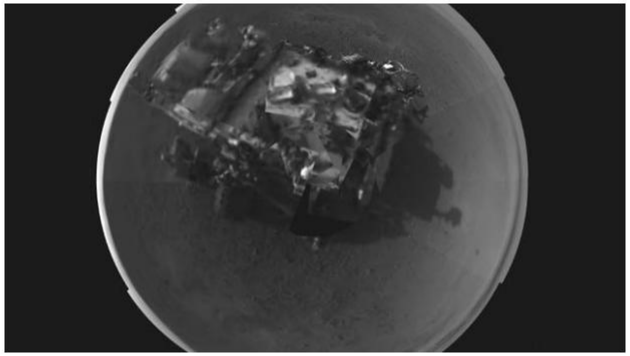
Cyber-physical Systems
(convergence of Computing, Communication and Control)

Mechatronics Applications

- Smart consumer products: home automation and security, microwave oven, toaster, dish washer, laundry washer-dryer, climate control,..
- Medical: implant-devices, assisted surgery, body area networks (BANs),..
- Defense: unmanned air, ground and undersea vehicles, smart munitions, jet engines,…
- Manufacturing: robotics, machines, processes, etc.
- Automotive: climate control, generation II antilock brake systems, active suspension, cruise control, air bags, engine management, safety, navigation, tele-operation, tele-diagnosis, backup collision sensing, rain sensing, etc.
- Cyber-physical Networked Systems: distributed robotics, tele-robotics, intelligent highways, smart grid, smart buildings, etc.

Examples of Mechatronic Systems -1

Computer disk drive

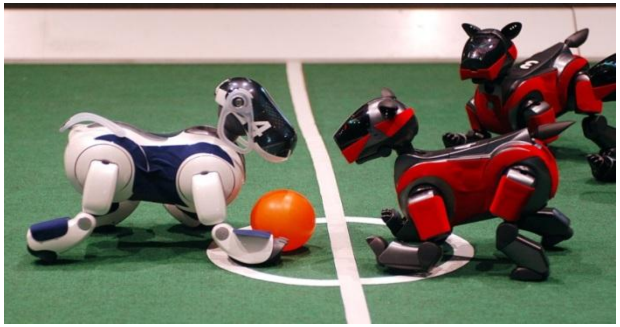


Mars Curiosity Rover

Asimo Humanoid (Honda)

Office Copier

Qurio Humanoid



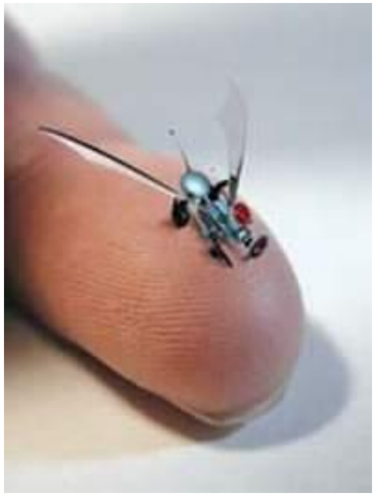
Robocup Team

Examples of Mechatronic Systems -2


Aviation

Flying UAV


Underwater Robot


Micro Robot

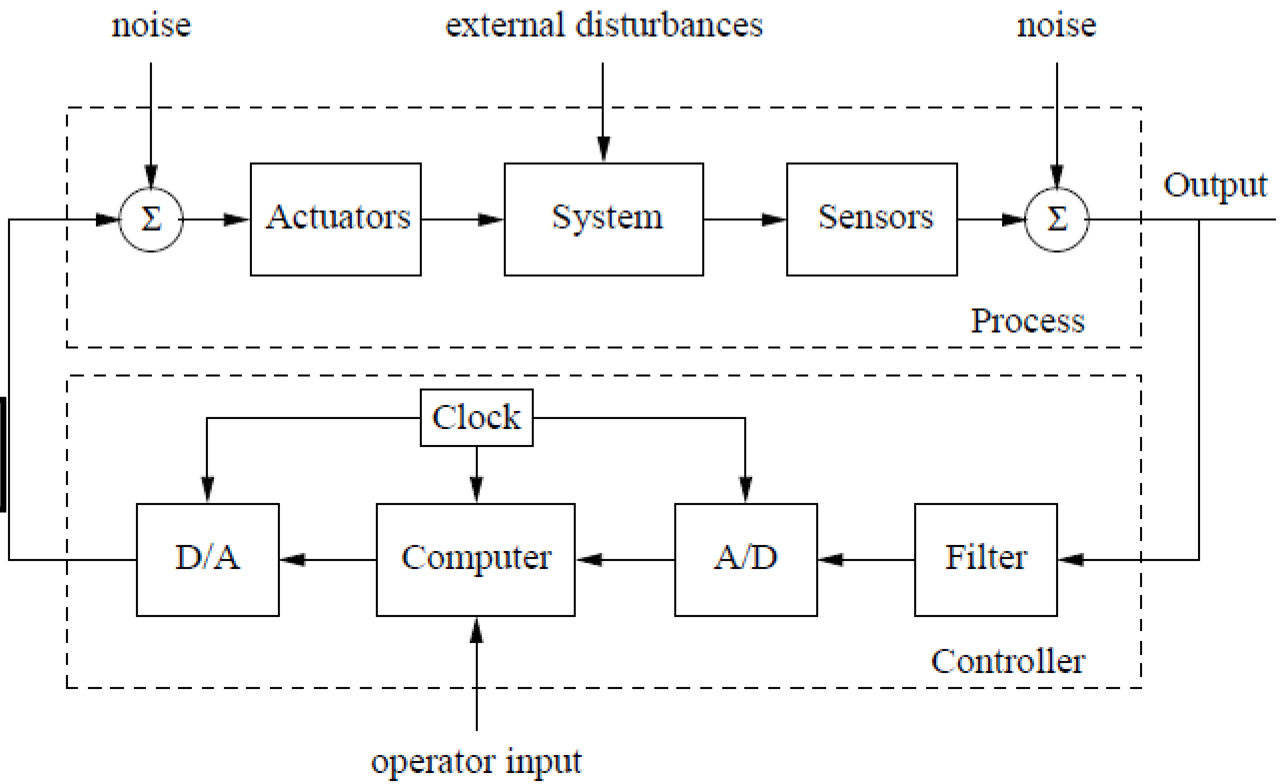

HEXAPOD Robot


Big Dog Robot

Control History Perspective

- First generation: Analog Control (prior to 1960)
  - – Technology: Feedback amplifiers and Proportional-Integral-Derivative (PID) controllers
  - – Theory: Frequency domain analysis
  - – Tools: Bode Plot, Root Locus, Nyquist Plot, Nichols Chart
- Second generation: Digital Control
  - – Around 1960-1970
  - – Technology: Digital computers and embedded microprocessors
  - – Theory: State-space design
  - – Real-Time Scheduling
- Third generation: Networked Control (Cyberphysical Systems .. phrase coined around 2006)
  - – Convergence of computing, communication and control throughout 1990s and beyond
  - – Theory: Hybrid systems, Formal methods, …
  - – Technology: Wireless and wired networks for remote sensing and actuation
  - – Numerous applications: Zero-accident highways, smart grid, tele-operation rooms, aerospace and transportation, robotics and intelligent machines, health,…..

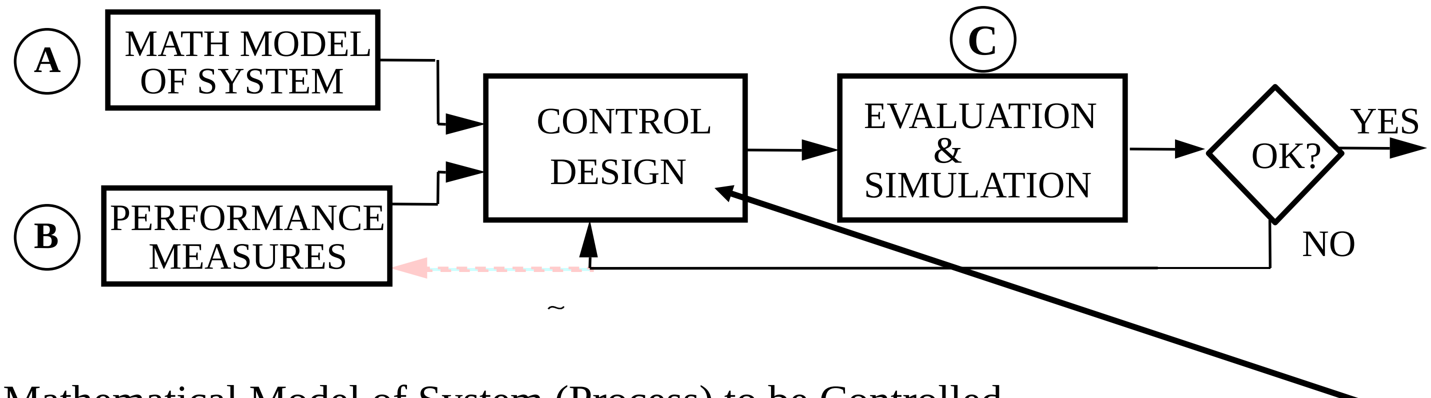
Feedback Control System



Adapted from
Astrom & Murray, 2011

- Process = actuators + dynamic system + sensors
- Noise and external disturbances perturb the dynamics of the process
- Controller = pre-filter + A/D + computer + D/A+ system clock
- Operator input = reference signal (set point, desired input)

Control System Design Process



Focus of the course

A) Mathematical Model of System (Process) to be Controlled

B) Performance Measures and Concerns

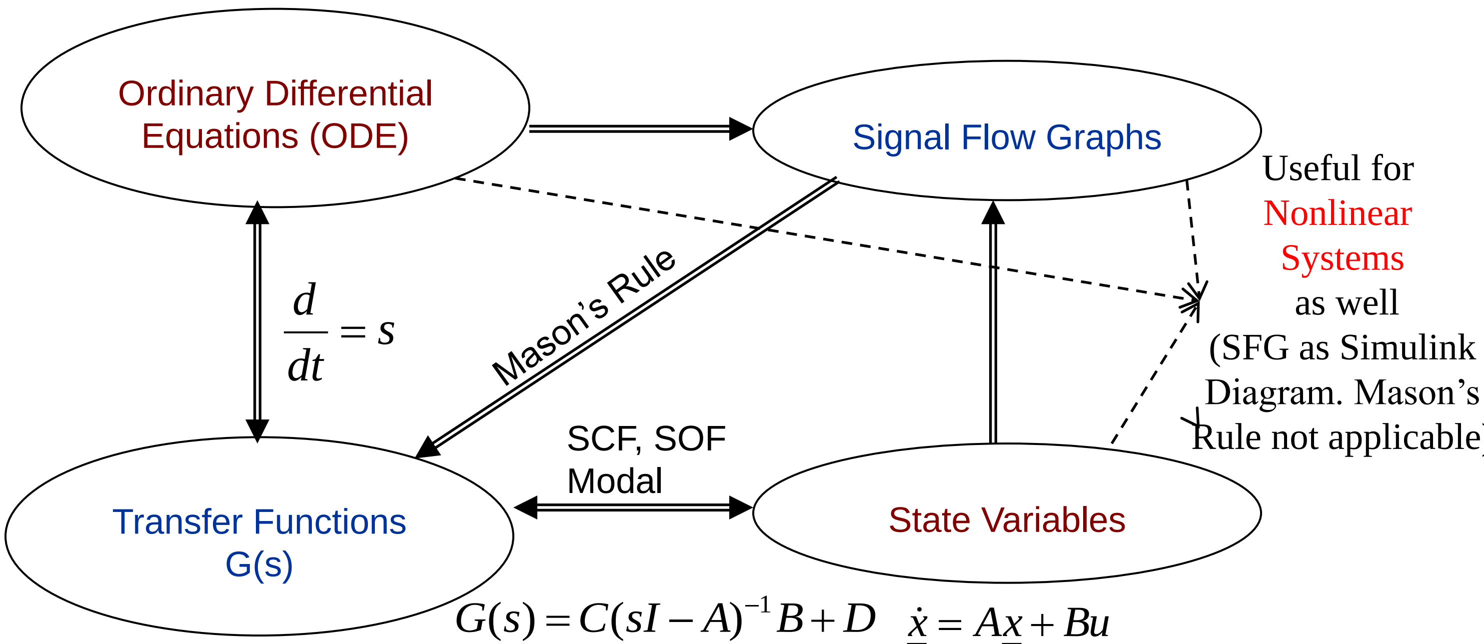
Mathematical criteria that are driven by customer's qualitative/quantittaive specifications for behavior of the closed-loop (feedback) system

(1) Stability of the closed-loop system
(2) Steady-state accuracy, Integral absolute error (IAE), integral square error (ISE)
(3) Speed of response/transient
(4) Sensitivity/robustness

C) Evaluation and Simulation

- Computer-aided Engineering (CAE) software tools such as MATLAB/Simulink,LabVIEW
- Hardware-in-the-loop simulation

Relationships among LTI Modeling Techniques



$$G(s) = C(sI - A)^{-1}B + D$$

$$\dot{\underline{x}} = A\underline{x} + Bu$$

$$y = C\underline{x} + Du$$

- Physical Variables
- Standard Controllable form (SCF)
- Standard Observable form (SOF)
- Modal (Diagonal) form

LTI: Linear Time-invariant

Math Modeling involves Diverse Disciplines - 1

- Classic Book
  - R. H. Cannon. *Dynamics of Physical Systems*. Dover, 2003 or McGraw-Hill, 1967.
- Mechanics
  - V. I. Arnold. *Mathematical Methods in Classical Mechanics*. Springer, 1978.
  - H. Goldstein. *Classical Mechanics*. Addison-Wesley, Cambridge, MA, 1953.
- Thermal
  - H. S. Carslaw and J. C. Jaeger. *Conduction of Heat in Solids*. Clarendon Press,1959.
- Fluids
  - J. F. Blackburn, G. Reethof, and J. L. Shearer. *Fluid Power Control*. MIT Press,1960.
- Vehicles
  - M. A. Abkowitz. *Stability and Motion Control of Ocean Vehicles*. MIT Press, 1969.
  - J. H. Blakelock. *Automatic Control of Aircraft and Missiles*. Addison-Wesley, 1991.
  - J. R. Ellis. *Vehicle Handling Dynamics*. Mechanical Engineering Pubs., London,1994.
  - U. Kiencke and L. Nielsen. *Automotive Control Systems: For Engine, Driveline, and Vehicle*. Springer, Berlin, 2000.

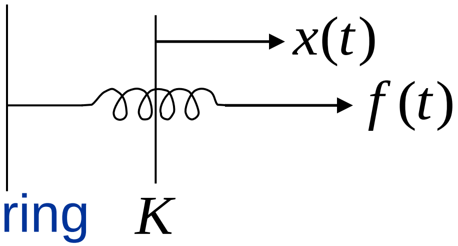
Math Modeling involves Diverse Disciplines - 2

- Robotics
  - M. W. Spong & M. Vidyasagar, *Dynamics and Control of Robot Manipulators*. Wiley, ‘89.
  - R. M. Murray, Z. Li, and S. S. Sastry. *A Mathematical Introduction to Robotic Manipulation*. CRC Press, 1994.
- Power Systems
  - P. Kundur. *Power System Stability and Control*. McGraw-Hill, New York, 1993.
- Acoustics
  - L. L. Beranek. *Acoustics*. McGraw-Hill, New York, 1954.
- Micromechanical Systems
  - S. D. Senturia. *Microsystem Design*. Kluwer, Boston, MA, 2001.
- Biological Systems
  - J. D. Murray. *Mathematical Biology*, Vols. I and II. Springer-Verlag, 2004.
  - H. R. Wilson. *Spikes, Decisions, and Actions: The Dynamical Foundations of Neuroscience*. Oxford University Press, Oxford, UK, 1999.

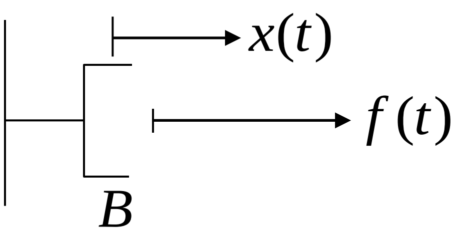
Force-Voltage Analogy for Translational Systems

Key Mechanical Elements:

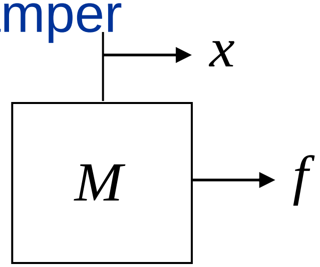
- spring
- viscous damper
- mass



spring



damper



mass

$$f = kx(t) = k\int v\,dt \Leftrightarrow V = q/c = \frac{1}{c}\int i\,dt$$

$$f = Bv = B\frac{dx}{dt} \Leftrightarrow V = Ri$$

$$f = Ma = M\frac{dv}{dt} \Leftrightarrow V = L\frac{di}{dt}$$

Analogy

force $= f \leftrightarrow$ voltage $= V$

position $= x \leftrightarrow$ charge $= q$

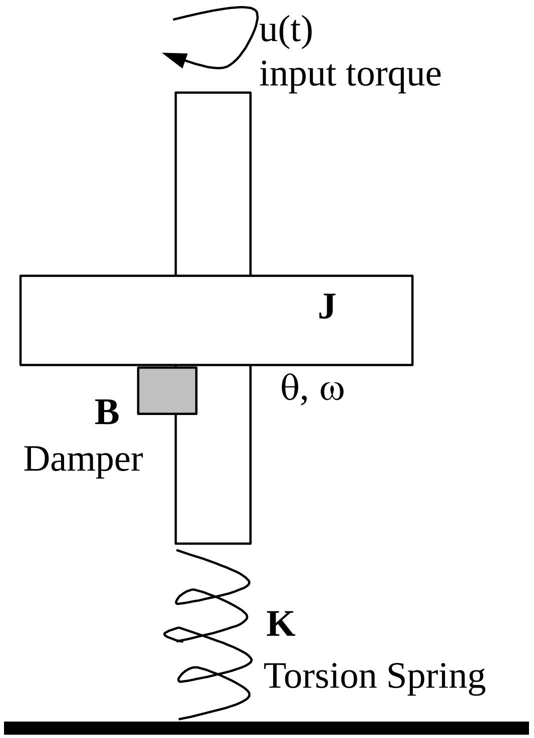
velocity $= v \leftrightarrow$ current $= i$

mass $= M \leftrightarrow$ inductor $= L$

damper $= B \leftrightarrow$ resistor $= R$

spring $= K \leftrightarrow$ capacitor $= \frac{1}{C}$

Torque-Voltage Analogy for Rotational Systems



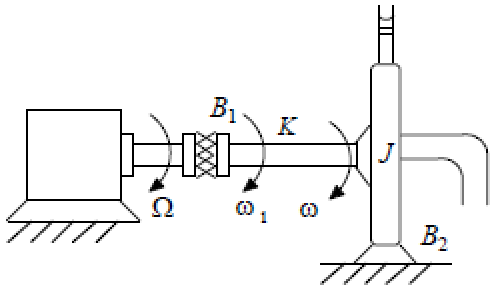
$$T = K\theta = K\int \omega \, dt \Leftrightarrow V = {}^{q}\!/\!{}_{C} = \frac{1}{C}\int i \, dt$$

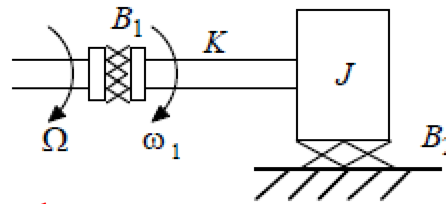
$$T = B\dot{\theta} = B\omega \quad \Leftrightarrow V = Ri$$

$$T = J\ddot{\theta} = J\dot{\omega} \Leftrightarrow V = L\frac{di}{dt}$$

- **Analogy**
  - Torque $= T \leftrightarrow$ Voltage $= V$
  - Displacement $= \theta \leftrightarrow$ Charge $= q$
  - Velocity $= \omega \leftrightarrow$ Current $= i$
  - Moment of inertia $= J \leftrightarrow$ Inductor $= L$
  - Damper $= B \leftrightarrow$ Resistor $= R$
  - Spring $= K \leftrightarrow$ Capacitor $= 1/C$

Diesel Engine Driving a Pump





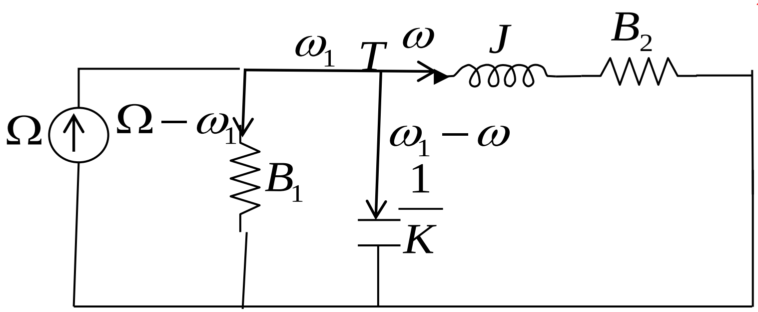
Input: Ω
Output: ω

Alternate States:

$(\omega, \theta_1 - \theta) = (\omega, x)$

$\dot{\omega} = -\frac{B_2}{J}\omega + K\,x$

$\dot{x} = \Omega - \omega - \frac{K}{B_1}x$



Electrical Analog

- **System Equations**

States :

Torque in the spring, T

Angular velocity of pump, ω

$$T = B_1(\Omega - \omega_1) = K(\theta_1 - \theta) = J\dot{\omega} + B_2\omega$$

$$\Rightarrow \dot{\omega} = -\frac{B_2}{J}\omega + \frac{1}{J}T$$

$$\Rightarrow \dot{T} = K(\omega_1 - \omega)$$

Also, since $\omega_1 = \Omega - \frac{1}{B_1}T$, *we have*

$$\dot{T} = -K\omega - \frac{K}{B_1}T + K\Omega$$

$$\begin{bmatrix}\dot{\omega}\\ \dot{T}\end{bmatrix} = \begin{bmatrix}-\frac{B_2}{J} & \frac{1}{J}\\ -K & -\frac{K}{B_1}\end{bmatrix}\begin{bmatrix}\omega\\ T\end{bmatrix} + \begin{bmatrix}0\\ K\end{bmatrix}\Omega$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\omega\\ T\end{bmatrix}$$

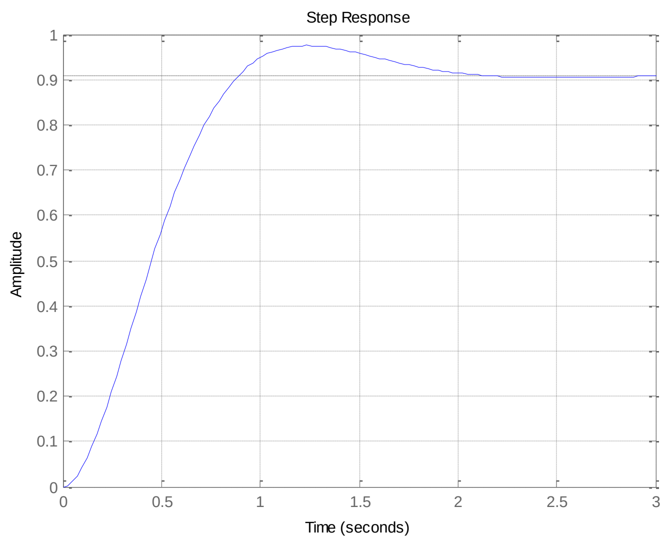
Draw Signal Flow Graph and Compute Transfer Function, Steady state gain? System type?

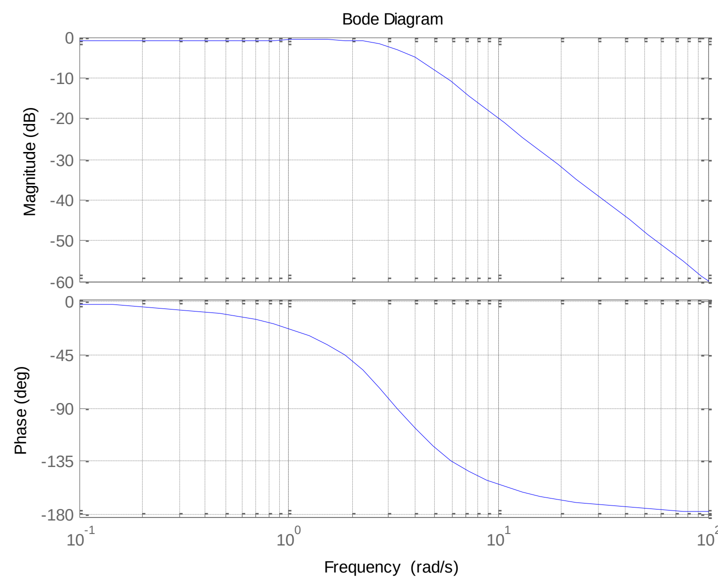
Analysis of Engine Driving a Pump

$K = 20 N.m/rad;$
$J = 2 kg.m^2/rad^2;$
$B_1 = 5 N.m.sec/rad;$
$B_2 = 0.5 N.m.sec/rad$

$$\begin{bmatrix}\dot{\omega}\\ \dot{T}\end{bmatrix} = \begin{bmatrix} -\frac{B_2}{J} & \frac{1}{J} \\ -K & -\frac{K}{B_1}\end{bmatrix}\begin{bmatrix}\omega\\ T\end{bmatrix} + \begin{bmatrix}0\\ K\end{bmatrix}\Omega \Rightarrow \begin{bmatrix}\dot{\omega}\\ \dot{T}\end{bmatrix} = \begin{bmatrix} -0.25 & 0.5 \\ -20 & -4\end{bmatrix}\begin{bmatrix}\omega\\ T\end{bmatrix} + \begin{bmatrix}0\\ 20\end{bmatrix}\Omega$$

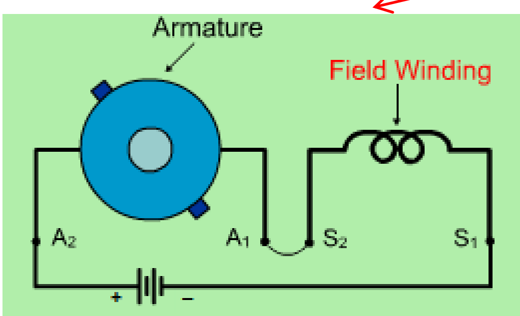
$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\omega\\ T\end{bmatrix} \Rightarrow G(s) = \frac{10}{s^2 + 4.25s + 11} \Rightarrow \omega_n = 3.32; \zeta = 0.64; dc\ gain = \frac{10}{11}$$

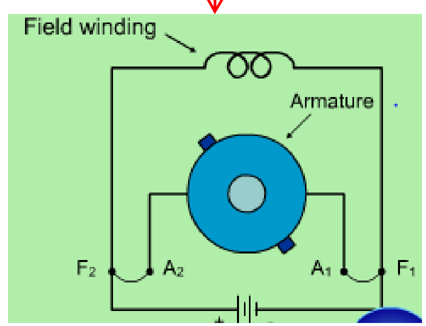


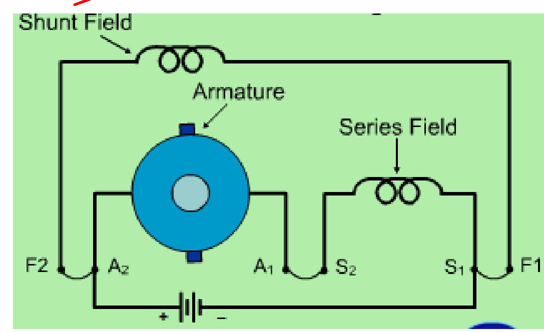


DC Motors

- DC Motors convert DC energy into mechanical energy $\rightarrow$ used in disk drives, robots, tape transport mechanisms
- Nice animation at:
  - http://www.wisc-online.com/objects/ViewObject.aspx?ID=IAU13208
  - http://www.wisc-online.com/objects/ViewObject.aspx?ID=IAU11508
  - http://www.wisc-online.com/objects/ViewObject.aspx?ID=IAU13708
- Torque of a DC motor: $T_m = K i_F i_A;\ i_F = Field\ \ current;\ i_A = Armature\ current$
  - Field controlled: i_A constant; i_F controlled; $\Rightarrow T_m = K_F i_F$
  - Armature controlled: i_F constant; i_A controlled; $\Rightarrow T_m = K_T i_A$
- Three types of DC motors: series wound, shunt and compound





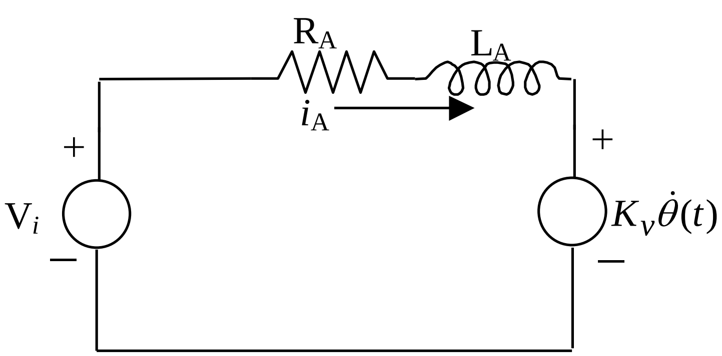


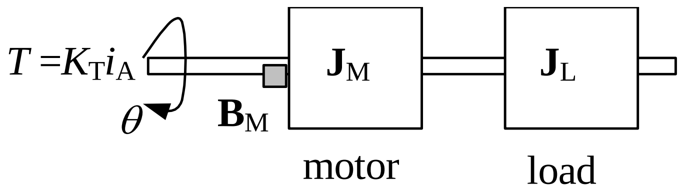
Armature Controlled DC Motors (Fixed Field)

- Fixed field is created by permanent magnets surrounding the armature
- Applying an input voltage to the armature circuit causes a current in the coils of the armature
- This generates a magnetic field which is repelled by the permanent field causing the motor to spin
- The torque generated is proportional to the armature current

- Modeling a DC Motor

Electro-mechanical system





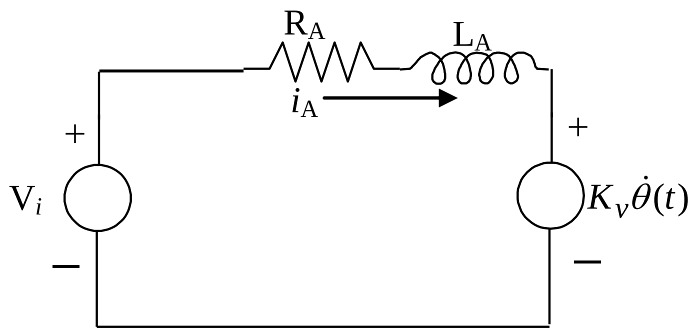
$K_v\dot{\theta}(t) =$ Reverse voltage; opposes input voltage V_i

Armature

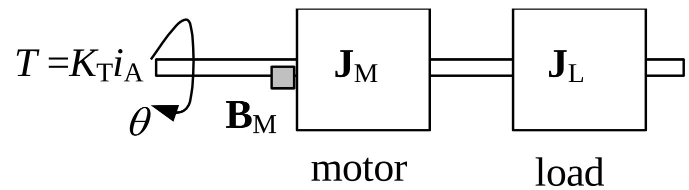
Mechanical Action

TF of an Armature Controlled DC Motor

- ***Modeling a DC Motor***



Armature



Mechanical Action

- ***Computing $\theta(s)/V_i(s)$***

Armature Equation:

$$v_i = R_A i_A + L_A \frac{di_A}{dt} + K_v \dot{\theta}(t)$$

$$\Rightarrow V_i(s) = R_A I_A(s) + sL_A I_A(s) + sK_v \theta(s) \quad (1)$$

Mechanical Dynamics Equation:

$$(J_M + J_L)\ddot{\theta}(t) = K_T i_A - B_M \dot{\theta}(t)$$

$$\Rightarrow (J_M + J_L)s^2\theta(s) = K_T I_A(s) - sB_M\theta(s)$$

$$\Rightarrow I_A(s) = \left(\frac{(J_M + J_L)}{K_T}s^2 + s\frac{B_M}{K_T}\right)\theta(s) \quad (2)$$

Substitute (2) into (1) and solve for $\theta(s)/V_i(s)$

$$\frac{\theta(s)}{V_i(s)} = \frac{K_T}{s\left(L_A(J_M + J_L)s^2 + (R_A(J_M + J_L) + L_A B_M)s + R_A B_M + K_T K_V\right)}$$

SS and SFG of an Armature Controlled DC Motor

Motor Equations:

$$v_i = R_A i_A + L_A \frac{di_A}{dt} + K_v \dot{\theta}(t)$$

$$\Rightarrow \frac{di_A}{dt} = v_i \frac{1}{L_A} - i_A \frac{R_A}{L_A} - \dot{\theta}(t) \frac{K_v}{L_A}$$

$$(J_M + J_L)\ddot{\theta}(t) = K_T i_A - B_M \dot{\theta}(t)$$

$$\Rightarrow \ddot{\theta}(t) = i_A \frac{K_T}{(J_M + J_L)} - \dot{\theta}(t) \frac{B_M}{(J_M + J_L)}$$

State Space:

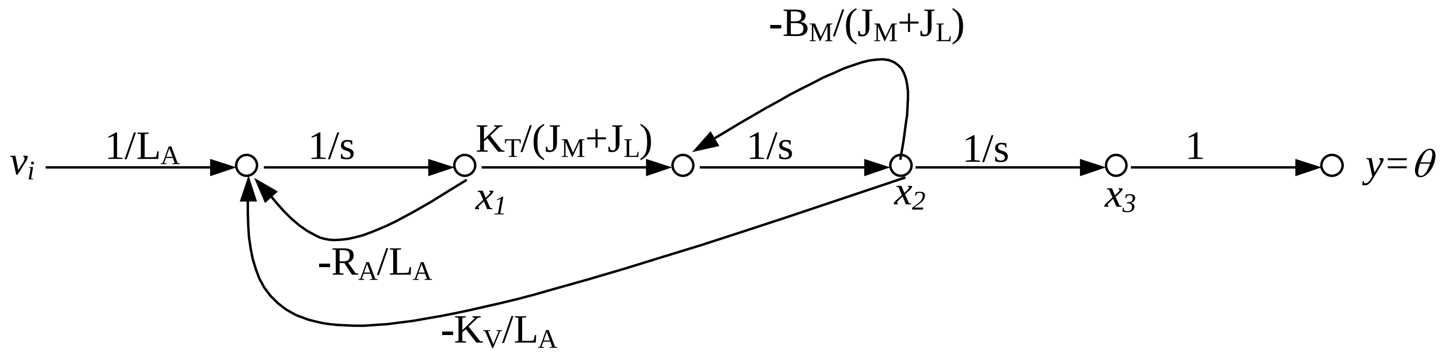
Declare states as $x_1 = i_A,\ x_2 = \dot{\theta} = \omega,\ x_3 = \theta$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -R_A/L_A & -K_v/L_A & 0 \\ K_T/(J_M + J_L) & -B_M/(J_M + J_L) & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1/L_A \\ 0 \\ 0 \end{bmatrix} v_i$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$R_A = 1\Omega;\ L_A = 0.001H;\ K_T = 5N-m/A;$
$K_v = 5volt-\sec/rad;\ B_M = 20kg/m/\sec;$
$(J_M + J_L) = 1N.m.\sec^2/rad$

SFG:



Steady-state Characteristics

Steady-state Equations:

$$\dot{\theta} = \omega = \frac{v_i - R_A i_A}{K_v}$$

$$T_m = K_T i_A = K i_f i_A$$

$$K_T = K_v = K i_f$$

$$\Rightarrow \omega = \frac{v_i}{K i_f} - \frac{R_A}{(K i_f)^2} T_m$$

$$T_m = 0 \Rightarrow No-load\ speed \Rightarrow \omega_0 = \frac{v_i}{K i_f}$$

$$\omega = 0 \Rightarrow stalling\ torque \Rightarrow T_S = \frac{K i_f v_i}{R_A} \Rightarrow \frac{\omega}{\omega_0} + \frac{T_m}{T_S} = 1$$

Speed Control Techniques:

- Armature resistance, R_A
- Field current (field flux), i_F
- Armature voltage, v_i

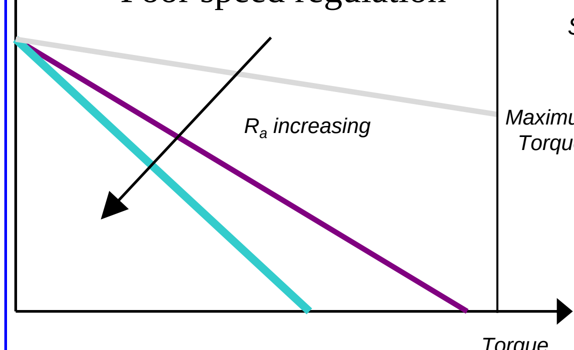
$Max\ power:$

$$P = T_m \omega = T_s (1 - \frac{\omega}{\omega_0})\omega$$

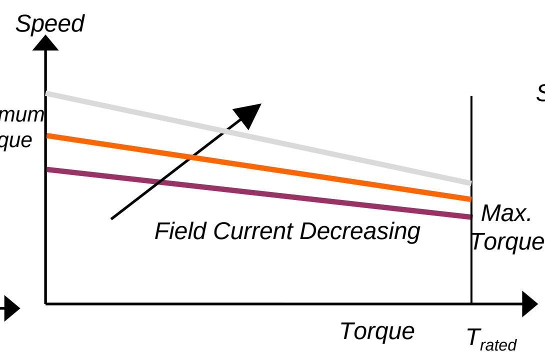
$$Max.\ at\ \omega = \frac{\omega_0}{2}$$

$$\Rightarrow P_{max} = \frac{T_s \omega_0}{4}$$

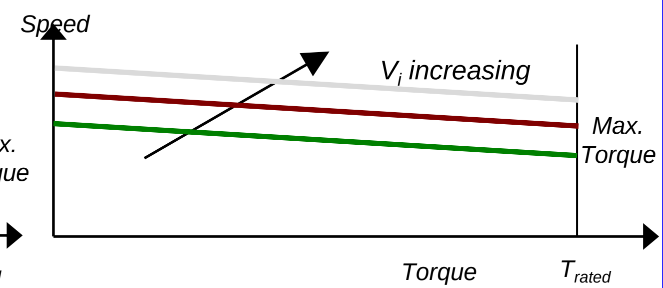
- Power loss in Ra
- Does not maintain max. torque capability
- Poor speed regulation



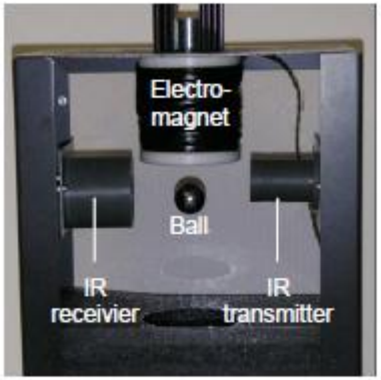
- Slow transient response
- Does not maintain max. torque capability



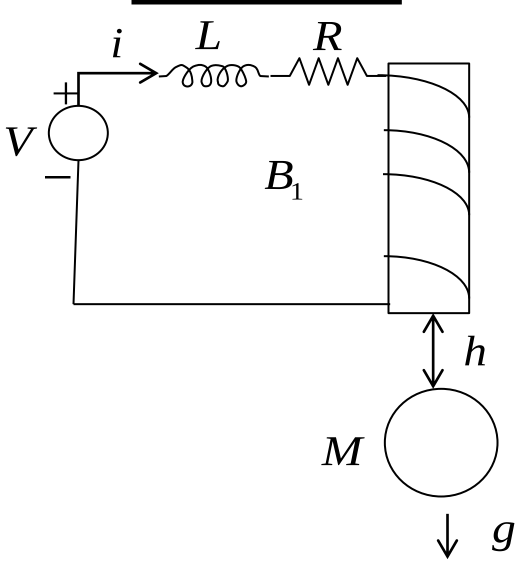
- good speed regulation
- maintains max. torque capability



Magnetic Levitation - 1



Adapted from Murray, 2004



$M = Mass\ of\ Ball = 0.05kg$

$h = Distance\ of\ Ball\ from\ Magnet$

$K = Coefficient\ in\ N\text{-}m/Amp^2 = 0.0001$

$L = Inductance\ in\ H = 0.01H$

$R = Resitance\ in\ \Omega = 1\ \Omega$

- **Nonlinear System Equations**

See Kuo's Book

$$M\ddot{h} = Mg - K\frac{i^2}{h^2}$$

$$V = L\frac{di}{dt} + Ri$$

$$\Rightarrow x_1 = h; x_2 = \dot{h}; x_3 = i$$

$$\dot{x}_1 = x_2; \dot{x}_2 = g - \frac{K}{M}\frac{x_3^2}{x_1}$$

$$\dot{x}_3 = -\frac{R}{L}x_3 + \frac{1}{L}V$$

IR sensor measurement:

$y = ah + b$

Linearize around equilibrium point: $(x_{1e} = h_0 \quad x_{3e} = i_0 \quad V_0)$

$$x_{3e} = i_0 = \sqrt{\frac{Mgx_{1e}^2}{K}} = \sqrt{\frac{Mgh_0^2}{K}}$$

$V_0 = Ri_0$

Magnetic Levitation - 2

- **Linearized System Equations:**

$$\Rightarrow x_1 = x_{1e} + \delta x_1 = h_0 + \delta x_1; x_2 = \delta x_2; x_3 = x_{3e} + \delta x_3 = i_0 + \delta x_3$$

$$\delta \dot{x}_1 = \delta x_2$$

$$\delta \dot{x}_2 = \frac{2K}{M}\frac{x_{3e}^2}{x_{1e}^3}\delta x_1 - \frac{2Kx_{3e}}{Mx_{1e}^2}\delta x_3$$

$$\delta \dot{x}_3 = -\frac{R}{L}\delta x_3 + \frac{1}{L}\delta V$$

$$\delta y = a\delta x_1$$

M = Mass of Ball = 0.05kg

h = Distance of Ball from Magnet

K = Coefficient in N-m / Amp^2 = 0.0001

L = Inductance in H = 0.01H

R = Resistance in Ω = 1 Ω

Linearize around equilibrium point: $(x_{1e} = h_0 = 0.01m \quad x_{3e} = i_0)$

$$x_{3e} = i_0 = \sqrt{\frac{Mgx_{1e}^2}{K}} = \sqrt{\frac{Mgh_0^2}{K}} = 0.7004A$$

$$V_0 = Ri_0 = Rx_{3e} = 0.7004V$$

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \\ \delta\dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1962 & 0 & -28.1 \\ 0 & 0 & -100 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix} \delta V$$

$$\delta y = \begin{bmatrix} a & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix}$$

$$G(s) = \frac{\delta Y(s)}{\delta V(s)} = \frac{-2810a}{s^3 + 100s^2 - 1962s - 196,200}$$

$$= \frac{-2810a}{(s+100)(s^2 - 1962)}$$

Neglecting motor dynamics

$$G(s) \approx \frac{-28.1a}{(s^2 - 1962)}$$

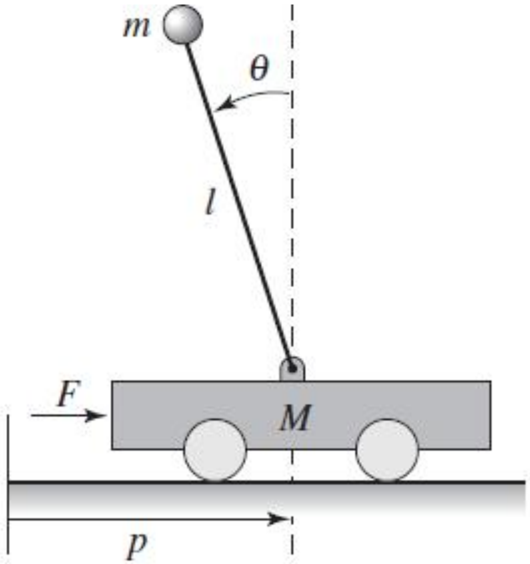
Open loop unstable

Inverted Pendulum

Astrom & Murray, 2011

(a) Segway

(b) Saturn rocket



(c) Cart–pendulum system

- **System Equations (prove in HW):** states $[p \quad \dot{p} \quad \theta \quad \dot{\theta}]$

$$\begin{bmatrix} F \\ 0 \end{bmatrix} = \begin{bmatrix} M+m & -ml\cos\theta \\ -ml\cos\theta & (J+ml^2) \end{bmatrix} \begin{bmatrix} \ddot{p} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} c\dot{p} + ml\sin\theta.\dot{\theta}^2 \\ \gamma\dot{\theta} - mgl\sin\theta \end{bmatrix}$$

$$outputs: \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} p \\ \theta \end{bmatrix}$$

M = Mass of base; m = mass of system to be balanced

J = Moment of intertia of system to be balanced

l = Distance from the base to the center of mass to be balanced

c, γ = Coefficients of viscous friction

g = Acceleration due to gravity

Inverted Pendulum

- **System Equations (prove in HW):** states $[p \quad \dot{p} \quad \theta \quad \dot{\theta}]$

$$Let\ M_t = M + m;\ J_t = J + ml^2$$

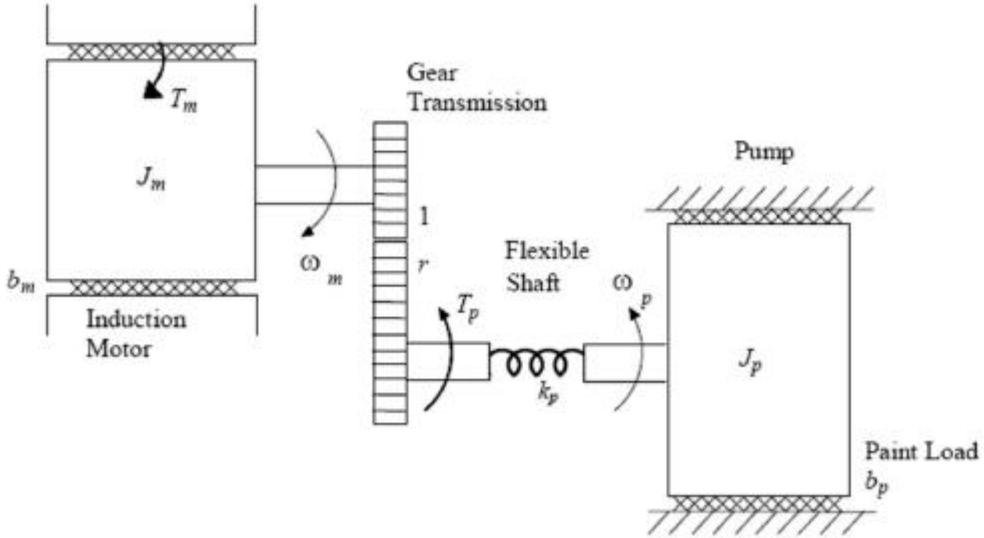
$$\frac{d}{dt}\begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{p} \\ \dot{\theta} \\ \dfrac{-ml\sin\theta.\dot{\theta}^2 + mg(ml^2/J_t)\sin\theta\cos\theta - c\dot{p} - (\gamma/J_t)ml\cos\theta.\dot{\theta} + F}{M_t - m(ml^2/J_t)\cos^2\theta} \\ \dfrac{-ml^2\sin\theta\cos\theta.\dot{\theta}^2 + M_t gl\sin\theta - cl\cos\theta\dot{p} - \gamma(M_t/m)\dot{\theta} + l\cos\theta F}{J_t(M_t/m) - ml^2\cos^2\theta} \end{bmatrix}; outputs: \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} p \\ \theta \end{bmatrix}$$

- **Linearized Equations:** $\theta \approx 0 \Rightarrow \sin\theta \approx \theta\ \&\ \cos\theta \approx 1\ and\ \dot{\theta}\ small \Rightarrow \dot{\theta}^2\ is\ negligible$

$$\frac{d}{dt}\begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & m^2l^2g/\mu & -cJ_t/\mu & -\gamma ml/\mu \\ 0 & M_tmgl/\mu & -cml/\mu & -\gamma M_t/\mu \end{bmatrix}\begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ J_t/\mu \\ ml/\mu \end{bmatrix} F;\ \mu = M_tJ_t - m^2l^2$$

$$outputs: \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix}$$

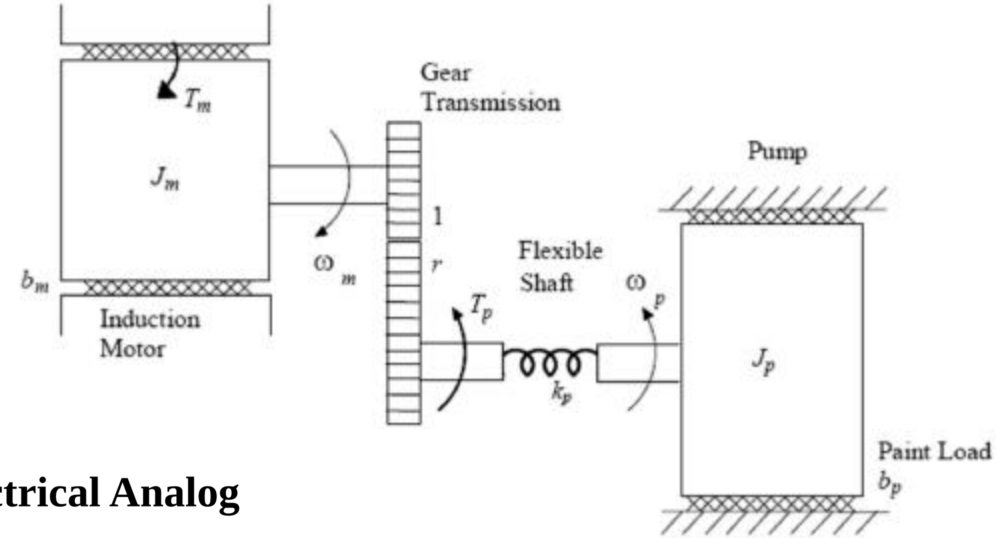
Spray Painting in an Automotive Plant -1



r : Gear Speed Ratio

- Inputs: Motor Torque, T_m is a function of line frequency, ω_0; Source voltage, v_ϕ *and* Motor speed, ω_m (which is a state variable) $\Rightarrow$ There is inherent feedback already!
- Output: Pump speed, ω_p
- Is the system linear? No, because T_m is a nonlinear function of $\{\omega_0, v_\phi \text{ and } \omega_m\}$
- How to get state equations? Gears are not perfect; They have efficiency, $\eta < 1$.
- How to get induction motor torque, T_m?

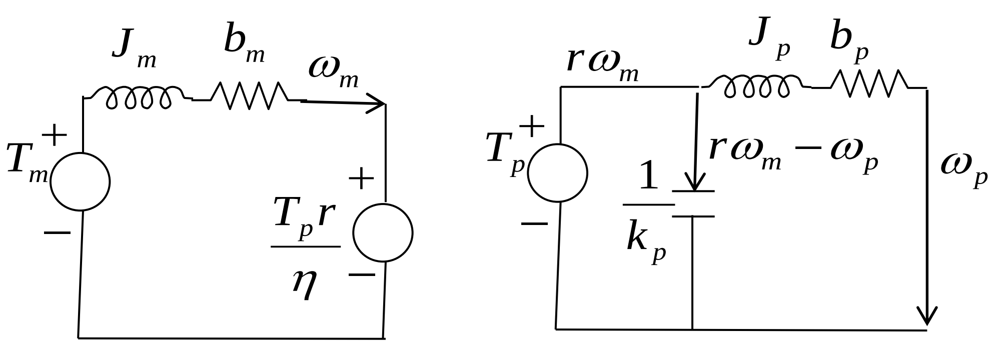
Spray Painting in an Automotive Plant -1



r : Gear Speed Ratio
$r < 1$ *if* $\omega_p < \omega_m$
Torque scales by $1/r$

$\omega_p = r\,\omega_m$ *in steady state*

- **Electrical Analog**



Electrical Analog

States : ω_m, T_p, ω_p

$$(1)\ J_m\dot{\omega}_m + b_m\,\omega_m + \frac{r}{\eta}T_p = T_m$$

$$\Rightarrow\ \dot{\omega}_m = -\frac{b_m}{J_m}\omega_m - \frac{r}{\eta J_m}T_p + \frac{1}{J_m}T_m$$

$$(2)\ \dot{T}_p = k_p(r\omega_m - \omega_p) = k_p r\omega_m - k_p\omega_p$$

$$(3)\ T_p = J_p\dot{\omega}_p + b_p\omega_p \Rightarrow \dot{\omega}_p = -\frac{b_p}{J_p}\omega_p + \frac{1}{J_p}T_p$$

T_m = Nonlinear $f(\omega_0, v_\phi, \omega_m)$

supply radian frequency

supply voltage

Magnetic Torque in an Induction Motor -1

- Very good animation and introduction to induction motors at
  - http://www.wisc-online.com/objects/ViewObject.aspx?ID=IAU10108
  - http://www.wisc-online.com/objects/ViewObject.aspx?ID=IAU13508

Synchronous speed (rpm):

$$n_{sync} = \frac{120 f_0}{P} = \frac{60\omega_0}{\pi P}$$

f_0 = supply *frequency* (Hz)

P = *Number of poles*

Slip

Rotor (Mechanical) speed (rpm) $= n_m$

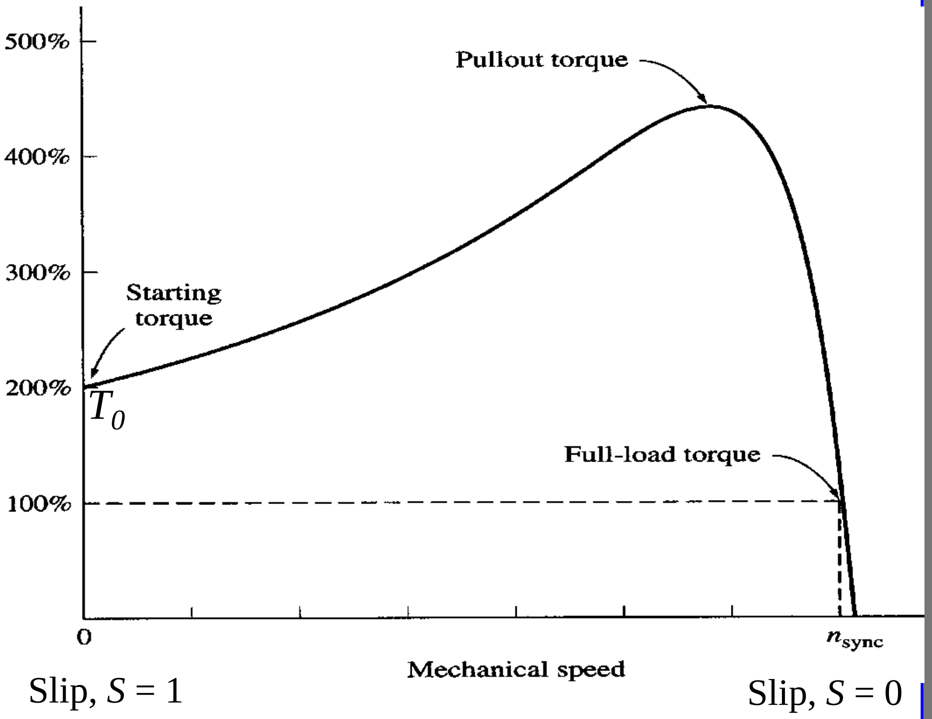
$$slip, S = \frac{n_{sync} - n_m}{n_{sync}} = \frac{\omega_0 - \omega_m}{\omega_0} = 1 - \frac{\omega_m}{\omega_0}$$

$$S = 0 \Rightarrow \omega_m = \omega_0 ; S = 1 \Rightarrow \omega_m = 0$$

slip speed $= sn_{sync}$

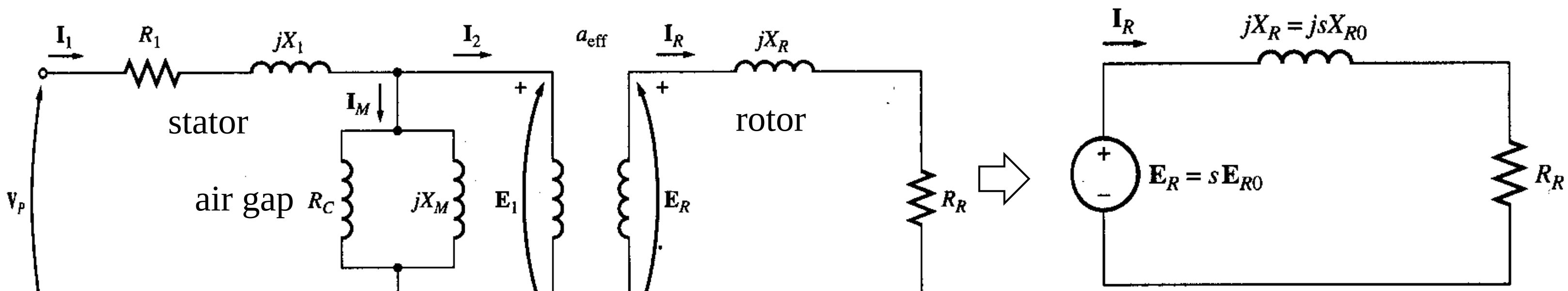
slip radian frequency $= s\omega_0$

= *radian freq. of voltage induced in rotor*



Magnetic Torque in an Induction Motor - 2

- Equivalent circuit of an induction motor. Assume 3 phases
  - Similar to a transformer except that the secondary windings rotate
  - Frequency of voltage induced in the rotor is $s\omega_0$
  - Voltage at s=0 is zero (no torque), Maximum at s=1
  - $\Rightarrow E_{R0} = E_1 N_R / N_S = E_1 / a_{eff};\ E_R = s\, E_{R0}$



Equivalent circuit

Rotor Equivalent circuit

$$I_R = \frac{sE_{R0}}{R_R + jsX_{R0}} = \frac{E_{R0}}{\frac{R_R}{s} + jX_{R0}}$$

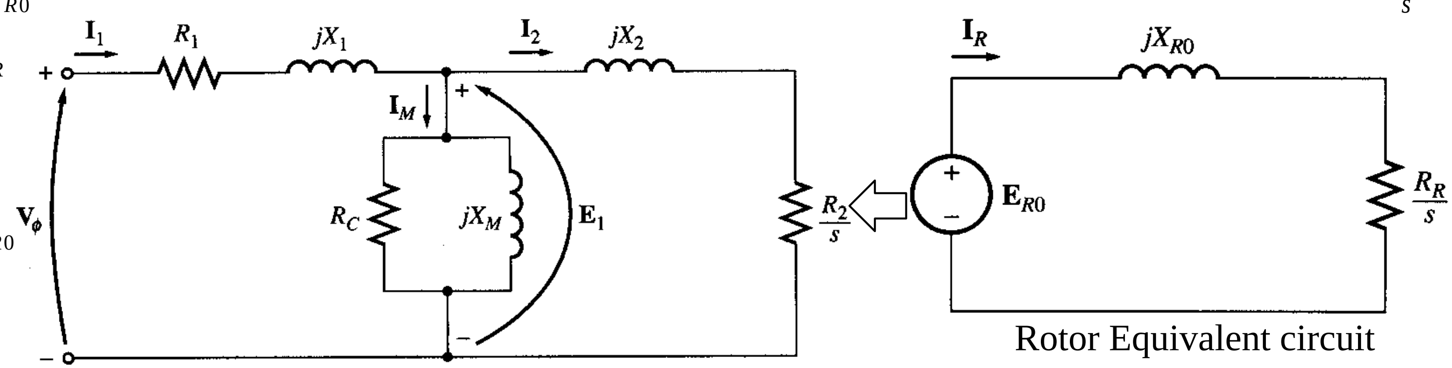
$$X_2 = a_{eff}^2 X_{R0}$$

$$R_2 = a_{eff}^2 R_R$$

$$I_2 = \frac{I_R}{a_{eff}}$$

$$E_1 = a_{eff} E_{R0}$$

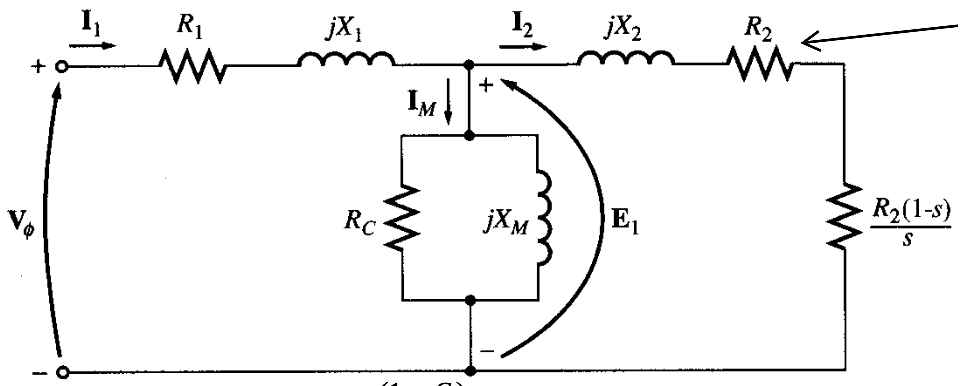
$$a_{eff} = \frac{N_S}{N_R}$$



Rotor Equivalent circuit

Magnetic Torque in an Induction Motor - 3

- Final Equivalent circuit of an induction motor. Assume 3 phases



Actual rotor resistance

Resistance equivalent to mechanical load

$$Mechanical\ \ Power, P_m = 3I_2^2 R_2 \frac{(1-S)}{S}$$

$$Neglecting\ \ stator\ impedance, I_2 = \frac{V_\phi}{\sqrt{\left(\frac{R_2}{S}\right)^2 + X_2^2}}$$

So, induction motors can be controlled by varying V_ϕ, ω_0 or R_2

$$\therefore P_m = \frac{3V_\phi^2 R_2(1-S)}{S\left[\left(\frac{R_2}{S}\right)^2 + X_2^2\right]} \Rightarrow T_m = \frac{P_m}{\omega_m} = \frac{3V_\phi^2 R_2 S}{\omega_0\left[R_2^2 + S^2 X_2^2\right]} \Rightarrow T_m = T_0 \frac{S\overbrace{\left[\frac{R_2^2}{X_2^2}+1\right]}^{q}}{\left[\frac{R_2^2}{X_2^2}+S^2\right]} = T_0 \frac{S\,q}{\left[\frac{R_2^2}{X_2^2}+S^2\right]}$$

$$where\ T_0 = T_m\big|_{S=1} = \frac{3V_\phi^2 R_2}{\omega_0\left[R_2^2 + X_2^2\right]} \Rightarrow T_m = \frac{T_0 q \omega_0(\omega_0 - \omega_m)}{q\omega_0^2 - \omega_m^2}\ \ for\ small\ S\ such\ that\ \omega_m^2 - 2\omega_0\omega_m \approx -\omega_m^2$$

Steady-state Operating Point & Linearization

- Steady-state operating point

$$\bar{T}_p = b_p \bar{\omega}_p ; \bar{\omega}_p = r\bar{\omega}_m ; \bar{T}_m = \left(b_m + \frac{r^2 b_p}{\eta} \right) \bar{\omega}_m = \bar{T}_0 \frac{q\bar{\omega}_0 (\bar{\omega}_0 - \bar{\omega}_m)}{\left(q\bar{\omega}_0^2 - \bar{\omega}_m^2 \right)}$$

$$\bar{T}_0 = \frac{\left(b_m + \frac{r^2 b_p}{\eta} \right) \bar{\omega}_m \left(q\bar{\omega}_0^2 - \bar{\omega}_m^2 \right)}{q\bar{\omega}_0 (\bar{\omega}_0 - \bar{\omega}_m)}$$

- State equations in matrix form

$$\begin{bmatrix} \dot{\omega}_m \\ \dot{T}_p \\ \dot{\omega}_p \end{bmatrix} = \begin{bmatrix} -b_m / J_m & -r/\eta J_m & 0 \\ K_p r & 0 & -K_p \\ 0 & 1/J_p & -b_p / J_p \end{bmatrix} \begin{bmatrix} \omega_m \\ T_p \\ \omega_p \end{bmatrix} + \begin{bmatrix} 1/J_m \\ 0 \\ 0 \end{bmatrix} T_m$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \omega_m \\ T_p \\ \omega_p \end{bmatrix}$$

Nonlinearity enters only through T_m !

Steady-state Operating Point & Linearization

- Linearized state equations: $\delta\omega_m = \omega_m - \bar{\omega}_m; \delta T_p = T_p - \bar{T}_p; \delta\omega_p = \omega_p - \bar{\omega}_p; \delta T_m = T_m - \bar{T}_m; \delta y = y - \bar{y}$

$$\begin{bmatrix}\delta\dot{\omega}_m \\ \delta\dot{T}_p \\ \delta\dot{\omega}_p\end{bmatrix} = \begin{bmatrix}(\partial T_m/\partial\omega_m - b_m)/J_m & -r/\eta J_m & 0 \\ K_p r & 0 & -K_p \\ 0 & 1/J_p & -b_p/J_p\end{bmatrix}\begin{bmatrix}\delta\omega_m \\ \delta T_p \\ \delta\omega_p\end{bmatrix} + \begin{bmatrix}\frac{\partial T_m/\partial V_\varphi}{J_m} & \frac{\partial T_m/\partial\omega_0}{J_m} & \frac{\partial T_m/\partial R_2}{J_m} \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\delta V_\varphi \\ \delta\omega_0 \\ \delta R_2\end{bmatrix}$$

$$\delta y = [0 \quad 0 \quad 1]\begin{bmatrix}\delta\omega_m \\ \delta T_p \\ \delta\omega_p\end{bmatrix}$$

- The required derivatives are evaluated at steady-state operating points

$$S = 1 - \frac{\omega_m}{\omega_0}; \bar{S} = 1 - \frac{\bar{\omega}_m}{\bar{\omega}_0}; \frac{\partial S}{\partial\omega_m} = -\frac{1}{\bar{\omega}_0}; \frac{\partial S}{\partial\omega_0} = \frac{\bar{\omega}_m}{\bar{\omega}_0^2}; \textit{Maximum Torque at } S_{T\max} = \frac{\bar{R}_2}{\bar{X}_2}$$

$$\frac{\partial T_m}{\partial\omega_m} = -\left(\frac{3\bar{V}_\phi^2\bar{R}_2}{\bar{\omega}_0^2}\right)\left(\frac{1-(\bar{S}/S_{T\max})^2}{\left(1+(\bar{S}/S_{T\max})^2\right)^2}\right) < 0 \ \textit{for} \ \bar{S} < S_{T\max}; \ \frac{\partial T_m}{\partial V_\phi} = \left(\frac{6\bar{V}_\phi\bar{S}}{\bar{\omega}_0\bar{R}_2\left(1+(\bar{S}/S_{T\max})^2\right)}\right) > 0;$$

$$\frac{\partial T_m}{\partial\omega_0} = \left(\frac{3\bar{V}_\phi^2}{\bar{\omega}_0^2\bar{R}_2\left(1+(\bar{S}/S_{T\max})^2\right)}\right)\left(\frac{\left(1-(\bar{S}/S_{T\max})^2\right)(1-\bar{S})}{\left(1+(\bar{S}/S_{T\max})^2\right)} - \bar{S}\left(1+\frac{2(\bar{S}/S_{T\max})^2}{\left(1+(\bar{S}/S_{T\max})^2\right)}\right)\right) > 0 \ \textit{for} \ \bar{S} < S_{T\max};$$

$$\frac{\partial T_m}{\partial R_2} = -\left(\frac{3\bar{V}_\phi^2\bar{S}}{\bar{\omega}_0\bar{R}_2\left(1+(\bar{S}/S_{T\max})^2\right)}\right)\left(\frac{\left(1-(\bar{S}/S_{T\max})^2\right)}{\left(1+(\bar{S}/S_{T\max})^2\right)}\right) < 0 \ \textit{for} \ \bar{S} < S_{T\max}$$

Math. Representation of (Sub)system Dynamics - 1

- We consider n state ($\underline{x}$), m input ($\underline{u}$), p output ($\underline{y}$) systems (vectors are columns)

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t), \underline{u}(t)); \quad \underline{x}(0) = \text{initial state}$$
$$\underline{y}(t) = \underline{g}(\underline{x}(t), \underline{u}(t)) \qquad (1.1)$$

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}; \quad \underline{u}(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix}; \quad \underline{y}(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_p(t) \end{bmatrix}$$

- The system dynamics are continuous, not discrete.
- Multivariable LTI: linear and time-invariant $\Rightarrow$ Superposition and constant parameter systems

(a) Linearized state space model

$$\delta\dot{\underline{x}}(t) = A\delta\underline{x}(t) + B\delta\underline{u}(t); \quad A = [a_{ij}] = [\frac{\partial f_i}{\partial x_j}]; B = [b_{ij}] = [\frac{\partial f_i}{\partial u_j}]$$

$$\delta\underline{y}(t) = C\delta\underline{x}(t) + D\delta\underline{u}(t); \quad C = [c_{ij}] = [\frac{\partial g_i}{\partial x_j}]; D = [d_{ij}] = [\frac{\partial g_i}{\partial u_j}] \qquad (1.2)$$

$A = n \times n$ System matrix

$B = n \times m$ Control matrix

$C = p \times n$ Output matrix

$D = p \times m$ I/O coupling matrix

Math. Representation of (Sub)system Dynamics - 2

(b) Transfer function representation

$s =$ Laplace transform variable

$$\Rightarrow \delta \underline{x}(s) = (sI - A)^{-1} B \delta \underline{u}(s) \Rightarrow \delta \underline{y}(s) = [C(sI - A)^{-1} B + D] \delta \underline{u}(s) = G(s) \delta \underline{u}(s) \quad (1.3)$$

$$\Rightarrow \delta y_k(s) = \sum_{j=1}^{m} g_{kj}(s) \delta u_j(s); k = 1, 2, .., p$$

$G(s) = [g_{kj}(s)] = p$ *by m transfer function matrix* (*TFM*)

$$g_{kj}(s) = \frac{\delta y_k(s)}{\delta u_j(s)} \Big|_{\delta u_i = 0; i \neq j}$$

$m = p = 1 \Rightarrow SISO$

$m = 1; p > 1 \Rightarrow SIMO$

$m > 1; p = 1 \Rightarrow MISO$

$m > 1; p > 1 \Rightarrow MIMO$

(c) Impulse response representation

$$G(s) = L[\text{Im}\, pulse\, \text{Re}\, sponse] = L[G(t)]$$

$$G(t) = \begin{cases} 0; t < 0 \\ Ce^{At} B + D\delta(t); t \geq 0 \end{cases}$$

Convolution Integral Formula :

$$\delta \underline{y}(t) = \int_0^t G(t - \sigma) \delta \underline{u}(\sigma) d\sigma = \int_0^t G(\sigma) \delta \underline{u}(t - \sigma) d\sigma = C \int_0^t e^{A(t-\sigma)} B \delta \underline{u}(\sigma) d\sigma + D \delta u(t) \quad (1.4)$$

Math. Representation of (Sub)system Dynamics - 3

(d) Variation 1: Disturbances, $\underline{d}$ and Measurement noise, $\underline{v}$

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t) + B_d\,\underline{d}(t)$$

$$\underline{y}(t) = C\underline{x}(t) + D\underline{u}(t) + D_d\,\underline{d}(t) + \underline{v}(t) \Rightarrow \underline{y}(s) = G(s)\underline{u}(s) + G_d(s)\underline{d}(s) + \underline{v}(s) \qquad (1.5)$$

(e) Variation 2: Descriptor representation

$$E\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t)$$

$$\underline{y}(t) = C\underline{x}(t) + D\underline{u}(t)$$

$$\Rightarrow \underline{y}(s) = G(s)\underline{u}(s) = [C(sE - A)^{-1}B + D]\underline{u}(s) \qquad (1.6)$$

Summary

1. What is Mechatronics?
2. Elements of Mechatronics
3. Mechatronics Applications
4. Example of Mechatronics Systems
5. Mathematical Modeling of Mechatronic Systems
6. General Representation of Systems