Lecture 14: Riccati Equation – Solution for Optimal Control Problem

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Outline of Lecture 14

- ❑ Continuous-time Linear Quadratic Regulator (LQR) problem
- ❑ Kleinman's algorithm for the Algebraic Riccati Equation (ARE)
  - properties
- ❑ Discrete-time LQR problem
- ❑ Schur method for solving the ARE

Infinite Horizon Continuous time LQR problem

- Problem
  - Consider a linear time-invariant system:

$$\dot{\underline{x}} = A\underline{x} + B\underline{u};\ \underline{x}(0) = \underline{x}_0$$

  - Want to find an optimal linear control law:

$$\underline{u} = -L\underline{x} \ni \int_0^\infty [\underline{x}^T Q\underline{x} + \underline{u}^T R\underline{u}]dt = J(u) \text{ is a minimum}$$

- Pick any gain $L_0 \ni A - BL_0 = \bar{A}_0$ is stable … will come back to the problem of picking L_0 later.

Recall that $J_0 = \underline{x}_0^T V_0 \underline{x}_0$ where V_0 satisfies the Lyapunov equation

$$0 = \bar{A}_0^T V_0 + V_0 \bar{A}_0 + Q + L_0^T R L_0 \qquad (1)$$

- Objective: want to find L^* that gives the smallest PD matrix P so that

$$J^* = \underline{X}_0^T P \underline{X}_0 \text{ is a minimum.}$$

Newton's Method for ARE - 1

- We seek an iterative process: $L_0 \to L_1 \to \ldots \to L_k \to \ldots \to L^*$
  - Find L_1 next that is better than L_0 , that is,

$$0 = \bar{A}_1^T V_1 + V_1 A_1 + Q + L_1^T R L_1 \qquad (2)$$

  - Note that

$$\bar{A}_0 = A - BL_0 = A - BL_1 + B(L_1 - L_0) = \bar{A}_1 - B\delta L; where\, \delta L = (L_0 - L_1)$$

  - Eq. (1) can be written in terms of $\bar{A}_1$ as:

$$0 = \bar{A}_1^T V_0 + V_0 \bar{A}_1 - \delta L^T B^T V_0 - V_0 B \delta L + Q + L_0^T R L_0$$

  - subtract Eq. (2) from Eq. (1), that is (1)-(2), to obtain:

$$0 = \bar{A}_1^T \delta V + \delta V \bar{A}_1 - \delta L^T B^T V_0 - V_0 B \delta L + L_0^T R L_0 - L_1^T R L_1; where\, \delta V = V_0 - V_1$$

  - this equation can be simplified to:

$$0 = \bar{A}_1^T \delta V + \delta V \bar{A}_1 + \delta L^T R \delta L - \delta L^T (B^T V_0 - R L_1) - (B^T V_0 - R L_1^T) \delta L$$

  - so, if $L_1 = R^{-1} B^T V_0$ then

$$0 = \bar{A}_1^T \delta V + \delta V \bar{A}_1 + \delta L^T R \delta L$$

Newton's Method for ARE - 2

$\Rightarrow \text{if } \bar{A}_1 \text{ is stable } \delta V \geq 0 \text{ or } V_1 \leq V_0$

$- \text{so, for every } \underline{x}_0, \text{ we have } \underline{x}_0^T V_0 \underline{x}_0 \geq \underline{x}_0^T V_1 \underline{x}_0$

$- \text{but, is } \bar{A}_1 \text{ stable? Yes!!}$

- from (1)

$$\bar{A}_1^T V_0 + V_0 \bar{A}_1 - \delta L^T R L_1 - L_1^T R \delta L + Q + L_0^T R L_0 = 0$$

$$or$$

$$\bar{A}_1^T V_0 + V_0 \bar{A}_1 + \delta L^T R \delta L + L_1^T R L_1 + Q = 0$$

can also see this form $V_0 = V_1 + \delta V$. Thus, $\bar{A}_1$ is stable by Lyapunov theorem, if $\bar{A}_0$ does.

- Continuing the iterative process with

$$L_{k+1} = R^{-1} B^T V_k, k = 0,1,2,\ldots,$$

$$\text{we obtain } V_{k+1} < V_k, k = 0,1,2,\ldots,$$

- $\{V_k\}$ are monotonically decreasing and bounded below by zero.

$$\Rightarrow \lim_{k \to \infty} V_k \to P \text{ exists and is PD}$$

Newton's Method for ARE - 3

- consequently, $L^* = R^{-1}B^T P$ is the converged gain

- Equation for P = cost matrix associated with L^*

$$(A - BR^{-1}B^T P)^T P + P(A - BR^{-1}B^T P) + Q + PBR^{-1}B^T P = 0$$

$$(or)\ A^T P + PA + Q - PBR^{-1}B^T P = 0$$

- The above Equation is called the Algebraic Ricatti equation (ARE)
- Thus, P is the unique PD solution of ARE

- **LQR Result**: The optimal control is $\underline{u} = -L^* \underline{x}$ *where* $L^* = R^{-1}B^T P$, where P is the unique PD solution of ARE. Well known result in optimal control……Linear quadratic regulator (LQR) problem.

Kleinman's Algorithm

- Kleinman's Algorithm for ARE (Newton's Method)

$$\text{pick } L_0 \ni A_0 \text{ is stable (to be addressed later)}$$

$$\text{Do for } k = 0,1,2....$$

$$\text{solve } \bar{A}_k^T V_k + V_k \bar{A}_k + Q + L_k^T R L_k = 0$$

$$L_{k+1} = R^{-1} B^T V_k ; \bar{A}_{k+1} = A_k - B L_{k+1}$$

$$\text{check for convergence}$$

$$\text{if } tr\, \delta V \le TOL.tr\,(V_k),$$

$$\text{stop, found } P, L^*$$

$$\text{end if}$$

$$\text{end DO}$$

Quadratic Convergence - 1

1. $P \le V_k \; \forall k$ infact, can show quadratic convergence

$$\bar{A}_k^T V_k + V_k \bar{A}_k + Q + L_k^T R L_k = 0$$

$$\bar{A}^{*T} P + P\bar{A}^* + Q + L^{*T} R L^* = 0$$

$$\bar{A}_k^T (V_k - P) + (V_k - P)\bar{A}_k + (L_k - L^*)^T R (L_k - L^*) = 0$$

$$\Rightarrow (V_k - P) = \int_0^\infty e^{\bar{A}_k^T \sigma} (L_k - L^*)^T R (L_k - L^*) e^{\bar{A}_k \sigma} d\sigma$$

– But, know that

$$L_k = R^{-1} B^T V_{k-1} \text{ and } L^* = R^{-1} B^T P$$

$$\Rightarrow (V_k - P) = \int_0^\infty e^{\bar{A}_k^T \sigma} (V_{k-1} - P)^T B R^{-1} B^T (V_{k-1} - P) e^{\bar{A}_k \sigma} d\sigma$$

– Taking norms on both sides:

$$\| V_k - P \| \le \int_0^\infty \| e^{\bar{A}_k \sigma} \|^2 \| B R^{-1} B^T \| \, \| V_{k-1} - P \|^2 \, d\sigma$$

$$= \| V_{k-1} - P \|^2 . \| B R^{-1} B^T \| \int_0^\infty \| e^{\bar{A}_k \sigma} \|^2 d\sigma$$

Quadratic Convergence - 2

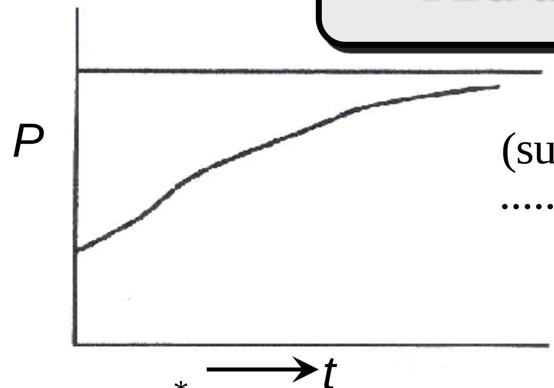
- So, in the limit we have quadratic convergence.
- Rate depends on $\left\| BR^{-1}B^T \right\|$ and on how stable $\bar{A}_k$ is.
- When V_k is far from P, we find linear convergence.

2. Algorithm is what you would get if you applied Newton's method to solve ARE directly.
   - But, we have also shown the need for stability of $\bar{A}_k$ and monotonicity.
3. P is the steady state solution of the Ricatti differential equation

$$\dot{P} = PA + AP + Q - PBR^{-1}B^T P$$

with $P(0) = $ arbitary PSD

Additional Insights - 1



(suggest using integration methods
..... but poor and lose PSD)

4. $u = -L^* \underline{x}$ is not just the optimal linear control law. It is the optimal control $\forall$ control law linear or nonlinear.
5. Scheme requires $\sim$ 10 iterations
   - $\approx 250n^3$ operations using iterative Lyapunov
   - $\approx 150n^3$ operations using Bartels-Stewart algorithm
6. Note that the Lyapunov equation must be solved to a greater accuracy.
   - If we want to solve the Lyapunov equation for P to an n-digit accuracy, we need to solve V_k to $(n+1)$-digit accuracy.
   - Usually 10^{-4} on trace (δV_k) is good enough convergence criterion.

Additional Insights - 2

7. If have "good" guess for L^*, say $\hat{L}$ (stabilizing) pick $L_0 = \hat{L}$
8. If have "good" guess for P, say, $\hat{P}$ pick $L_0 = R^{-1}B^T\hat{P}$, is stable.
9. If the process stops before convergence, have better L than when started.
10. If want to minimize $J = \lim_{T\to\infty} \left(\int_0^T (\underline{x}^T Q \underline{x} + \underline{u}^T R \underline{u})\, dt \right) = E(\underline{x}^T Q \underline{x} + \underline{u}^T R \underline{u})$

    for the linear stochastic system:

    $$\dot{\underline{x}} = A\underline{x} + B\underline{u} + E\underline{w}$$

    where $\underline{w}$ is zero mean white Gaussian noise vector with covariance matrix W.

- Then the cost can be rewritten as:

  $$J = E\{\underline{x}^T Q \underline{x} + \underline{u}^T R \underline{u}\} \text{ and } J^* = \mathrm{tr}(PEWE^T)$$

How to Pick Initial Gains? - 1

- Again same control law.
- This is the so called certainty equivalence(CE) property.

❑ One unanswered question: How to pick L_0?

- We know from Lecture 3 that

$$L_0 = B^T e^{A^T T}\left[\int_0^T e^{A\sigma} BB^T e^{A^T\sigma} d\sigma\right]^{\dagger} e^{AT} \text{ stabilizes } A - BL_0$$

- Replace $B \rightarrow \widetilde{B} = BR^{-1/2}$
- Since $R > 0$, $R^{-1/2}$ always exists

$$L_0 = R^{-1/2} B^T e^{A^T T}\left[\int_0^T e^{A\sigma} BR^{-1} B^T e^{A^T\sigma} d\sigma\right]^{\dagger} e^{AT}$$

stabilizes $A - BR^{-1/2} L_0$

or $L_0 = R^{-1} B^T e^{A^T T}\left[\int_0^T e^{A\sigma} BR^{-1} B^T e^{A^T\sigma} d\sigma\right]^{\dagger} e^{A^T T}$

stabilizes $A - BL_0\ \forall\ R > 0$

How to Pick Initial Gains? - 2

- Note that L_0 stabilizes A and is of the form $L_0 = R^{-1}B^T V_{-1}$, where

$$V_{-1} = e^{A^T T}\left[\int_0^T e^{A\sigma} B R^{-1} B^T e^{A^T \sigma} d\sigma\right]^{\dagger} e^{AT}$$

❑ The computation of V_{-1} proceeds as follows:

1. Compute $\int_0^T e^{A\sigma} B R^{-1} B^T e^{A^T \sigma} d\sigma = W(T)$......Lecture 3
2. Factor $W = \Gamma\Gamma^T$; $\Gamma = n \times p$; p = rank W
3. Compute $\Gamma^{\dagger}$. $W^{\dagger} = (\Gamma^{\dagger})^T \Gamma^{\dagger}$
4. $Z = \Gamma^{\dagger} e^{AT}$; $V_{-1} = Z^T Z$

How to Pick Initial Gains? - 3

- Note that in theory T is arbitrary. But, in practice, try to pick $T \ni V_{-1}$ is close to P … an open problem.
  - Usually, the following choices work:
    1. $T = 1$
    2. $T = 2/\|A\| \approx 2/|\lambda_{\max}(A)|$
    3. $T = 2/|\lambda_{avg}(\bar{A})| = 2n/|tr(\bar{A})|$
  - For 3), how to get $tr(\bar{A})$?

    assume $BR^{-1}B^T = S$ has an inverse.

    Note that

$$\bar{A}^T S^{-1} \bar{A} = (A^T - PS) S^{-1} (A - SP)$$
$$= A^T S^{-1} A - PA - A^T P + PSP = A^T S^{-1} A + Q$$

take $tr(\bar{A}^T S^{-1} \bar{A})$ to obtain:

$$tr(\bar{A}\bar{A}^T S^{-1}) = tr((AA^T + QS)S^{-1})$$

How to Pick Initial Gains? - 4

Use

$$tr(AA^T + QS) \text{ as an estimate of } tr(\bar{A}\bar{A}^T) = \sum_{i=1}^{n} \sigma_i^2,$$

where $\{\sigma_i\}$ are singular values of $\bar{A}$

further use

$$\frac{1}{n}\sqrt{tr(AA^T + QS)} \text{ as an estimate of } |\lambda_{avg}(\bar{A})|$$

$$\Rightarrow \; T = \frac{2n}{\sqrt{tr(AA^T + QS)}}$$

- If V_{-1} or L_0 fails to stabilize, double T and continue the process

Discrete ARE (DARE)

- Infinite Horizon Discrete LQ regulator problem
- Problem
  - Consider a discrete-time system:

    $$\underline{x}_{i+1} = \Phi \underline{x}_i + B\underline{u}_i$$

  - Find a linear feedback control law $\underline{u}_i = -L\underline{x}_i$ to minimize

    $$J = \sum_{i=0}^{\infty} [\underline{x}_i^T Q \underline{x}_i + \underline{u}_i^T R \underline{u}_i]$$

- We can develop an algorithm in parallel to the continuous case.
  - Suppose have gain L_k, then

    $$J_k = \underline{x}_0^T V_k \underline{x}_0$$

    where V_k satisfied the algebraic Riccati equation:

    $$V_k = \overline{\Phi}_k^T V_k \overline{A}^T \overline{\Phi}_k + Q + L_k^T R L_k$$

    where $\overline{\Phi}_k = \overline{\Phi} - BL_k$ stable

Newton's Mathod for DARE

- LQR Results: optimal control $\underline{u}_i^* = -(R + B^T PB)^{-1} B^T P \Phi \underline{x}_i$ (some times written as $-L\Phi\, \underline{x}_i$), where P is the unique PD solution of the discrete ARE

$$P = \Phi^T [P - PB(R + B^T PB)^{-1} B^T P]\Phi + Q$$
$$= \Phi^T P(I + SP)^{-1}\Phi + Q$$
$$= \Phi^T (P^{-1} + S)^{-1}\Phi + Q \quad \text{where } S = BR^{-1}B^T$$

- Iterative algorithm

Pick $L_0 \ni \bar{\Phi}_0$ is stable

Do for k=0,1,2,…

solve

$$V_k = \bar{\Phi}_k^T V_k \bar{\Phi}_k + Q + L_k^T R L_k$$
$$L_{k+1} = (R + B^T V_k B)^{-1} B^T V_k \Phi$$
$$\bar{\Phi}_{k+1} = \Phi - BL_{k+1} = (I + SV_k)^{-1}\Phi$$

check for convergence:

If $\text{tr}(\delta V_k) \le TOL.\text{tr}(V_k)$

stop. obtained $P=V_k$ and $L^*=L_{k+1}$

end if

end Do

Algorithmic Properties - 1

- Properties much the same as in the continuous case
  - Quadratic convergence
  - Takes ~ 10 iterations
  - $J(u^*) = \underline{x}_0{}^T P \underline{x}_0$.
  - Also, P is the steady state solution of the difference Riccati equation

    $$P_{i+1} = \Phi^T [P_i - P_i B(R + B^T P_i B)^{-1} B^T P_i] \Phi + Q$$

    where $[P_i - P_i B(R + B^T P_i B)^{-1} B^T P_i]$ is the update
  - Recall that this is similar to the update – propagate equation of Kalman filter with the associations:
    - $\Phi \rightarrow \Phi^T$
    - $B \rightarrow C^T$

Algorithmic Properties - 2

- If we are only interested in $P = \Phi^T P(I+SP)^{-1}\Phi+Q$, use

$$V_k = \bar{\Phi}_k^T V_k \bar{\Phi}_k + Q + L_k^T R L_k + \bar{\Phi}_k^T V_{k-1} S V_{k-1} \bar{\Phi}_k$$

where $\bar{\Phi}_k=(I+SV_{k-1})^{-1}\Phi$
and where V_{-1} is picked $\ni$ $\bar{\Phi}_0$ is stable.

- Initialization (Kleinman, IEEE Trans. On AC, June 1974)
  - $M > n$ (power of 2) arbitrary
  - $L_0 =(R+B^T V_{-1} B)^{-1} B^T V_{-1}\Phi$

$$V_{-1} = (\Phi^T)^M \left[\sum_{i=0}^{M-1} \Phi^k S (\Phi^T)^k \right]^{\dagger} \Phi^M$$

Schur Method for solving ARE

- Consider the continuous time *ARE*

  $A^TP+PA+Q\text{-}PSP=0\;;\;\; S=BR^{-1}B^T$

We can show that

$$\begin{bmatrix} \dot{\underline{x}}(t) \\ \dot{\underline{p}}(t) \end{bmatrix} = \begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} \underline{x}(t) \\ \underline{p}(t) \end{bmatrix} \text{ with } p(0) = P(0)x(0);\; \underline{p}(t) = costate$$

- These are the so-called two-point boundary value problem (TPBVP) equations
  - Then $p(t)=P(t)\,\underline{x}(t)$, where $P(t)$ satisfies the Riccati differential equation

    $$\dot{P} = A^TP + PA + Q - PSP$$

  - The matrix $Z = \begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix}$ is called the Hamiltonian

Hamiltonian Properties - 1

$$J^T Z^T J = -Z$$

where $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$

$$J^T Z J = -Z^T$$

$$\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}\begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}\begin{bmatrix} S & A \\ A^T & -Q \end{bmatrix} = \begin{bmatrix} -A^T & Q \\ S & A \end{bmatrix} = -Z^T$$

- Note ***tr (Z)*** **= 0** $\Rightarrow$ Eigen values are symmetrically disposed around the origin
  - Indeed if $\lambda_i(Z)$ is an Eigen value of Z, so is $-\lambda_i(Z)$
  - Furthermore, same multiplicity

Hamiltonian Properties - 2

- Suppose $\begin{pmatrix} \underline{a} \\ \underline{b} \end{pmatrix}$ is an Eigen vector of Z for λ_i

Then, $\begin{pmatrix} A & -S \\ -Q & -A^T \end{pmatrix}\begin{pmatrix} \underline{a} \\ \underline{b} \end{pmatrix} = \lambda_i \begin{pmatrix} \underline{a} \\ \underline{b} \end{pmatrix} \Rightarrow \begin{matrix} A\underline{a} - S\underline{b} = \lambda_i \underline{a} \\ -Q\underline{a} - A^T\underline{b} = \lambda_i \underline{b} \end{matrix}$

Note that

$$\begin{pmatrix} A^T & -Q \\ -S & -A \end{pmatrix}\begin{pmatrix} -b \\ a \end{pmatrix} = -\lambda_i \begin{pmatrix} -b \\ a \end{pmatrix} \Rightarrow \begin{matrix} -A^T\underline{b} - Q\underline{a} = \lambda_i \underline{b} \\ S\underline{b} - A\underline{a} = -\lambda_i \underline{a} \end{matrix}$$

So, $\begin{pmatrix} -b \\ a \end{pmatrix}$ is also an eigen value of Z^T

since $\lambda_i(Z) = \lambda_i(Z^T) \Rightarrow \lambda_i, -\lambda_i$ are eigen values of Z.

Schur Method for Solving ARE

- Find an orthogonal transformation Q (2n x 2n)

$$Q^T Z Q = \tilde{Z} = \begin{pmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} \\ 0 & \tilde{Z}_{22} \end{pmatrix}, \text{ where } \tilde{Z} \text{ is an upper Schur form (real)}$$

- Moreover, it is possible to arrange such that the real parts of the spectrum of $\tilde{Z}_{11}$ are negative, while those of $\tilde{Z}_{22}$ are positive

- Write Q such that it is conformal with Z.

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$$

Theory Behind Schur Method - 1

- Theorem:
  1. Q_{11} is invertible
  2. $P = Q_{21}Q_{11}^{-1}$, and P is symmetric PD matrix
  3. $\lambda_i\left(Z_{11}\right) = \lambda_i\left(A - BL^*\right) = \lambda_i\left(A - SP\right) =$ eigen values of the closed loop system

- Proof:

Let T be

$$T^{-1}ZT = \begin{pmatrix} -\Lambda & 0 \\ 0 & \Lambda \end{pmatrix}, \ \Lambda \text{ can be complex}$$

but, $\Lambda =$ diagonal matrix with postive real parts

then, $Z\begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} = -\begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix}\Lambda$ and $Z\begin{pmatrix} Q_{11} \\ Q_{21} \end{pmatrix} = \begin{pmatrix} Q_{11} \\ Q_{21} \end{pmatrix}\tilde{Z}_{11}$

Theory Behind Schur Method - 2

- Proof (contd.) :

$$\text{Let } \Gamma^{-1}\tilde{Z}_{11}\Gamma = -\Lambda$$

$$\Rightarrow Z\begin{pmatrix} Q_{11} \\ Q_{21} \end{pmatrix}\Gamma = \begin{pmatrix} Q_{11} \\ Q_{21} \end{pmatrix}\Gamma\Gamma^{-1}\tilde{Z}_{11}\Gamma = -\begin{pmatrix} Q_{11} \\ Q_{21} \end{pmatrix}\Gamma\Lambda \qquad (2)$$

$$\Rightarrow \begin{pmatrix} Q_{11} \\ Q_{21} \end{pmatrix}\Gamma = \begin{pmatrix} \mathrm{T}_{11} \\ \mathrm{T}_{21} \end{pmatrix}\mathrm{D}, \text{ where D is a diagonal matrix with } \pm 1\text{'s}$$

$$\begin{pmatrix} Q_{11} \\ Q_{21} \end{pmatrix} = \begin{pmatrix} \mathrm{T}_{11} \\ \mathrm{T}_{21} \end{pmatrix}\mathrm{D}\Gamma^{-1}$$

Since $\mathrm{T}_{21}\mathrm{T}_{11}^{-1}$ solves Ricatti equation, so does $Q_{21}Q_{11}^{-1}$. Why ?

- Theorem: **(*Potter, 1963*)** $T_{21}T_{11}^{-1}$ solves Riccati equation
- Proof:
  Let $G = A - SP$ = closed-loop matrix.

  then $PG = -(Q + A^T P)$

  Let U be the set of Eigen vectors of $\bar{A} = A - SP$.

Theory Behind Schur Method - 3

Then $U^{-1}GU = -\Lambda \Rightarrow GU = -U\Lambda$ (1)

$$PGU = -(Q + A^T P)U \quad (2)$$

Let $PU = V \Rightarrow PGU = -V\Lambda = -QU - A^T V$ (3)

Also, $GU = AU - SPU = AU - SV$ (4)

From (3) and (4), we have

$$\begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = -\begin{bmatrix} U \\ V \end{bmatrix} \Lambda$$

$\Rightarrow \begin{bmatrix} U \\ V \end{bmatrix}$ are the eigen vectors of Z corresponding to $-\Lambda$

$$\Rightarrow P = VU^{-1} = \mathrm{T}_{21}\mathrm{T}_{11}^{-1} = \mathrm{Q}_{21}\mathrm{Q}_{11}^{-1}$$

Why Schur & NOT Potter's Method?

- Computing via Eigen vector method is bad because the Eigen vector computation leads to numerical instabilities

- However, Schur method is OK because Q is orthogonal

- So, solve $PQ_{11} = Q_{21}$ via LU decomposition

$$Q_{11}^T P = Q_{21}^T$$

- Computational Load
  - Transform $Z \rightarrow$ upper Hessenberg: $5\,(2n)^3/3$
  - Upper Hessenberg $\rightarrow$ Upper Schur = $4\,\sigma(2n)^3 = 48n^3;\ \sigma = 1.5$
  - Solution $PQ_{11} = Q_{21}$ via LU decomposition $\Rightarrow 4n^3/3$

  $\Rightarrow$ So, total computational load = $63\,n^3$

  $\Rightarrow$ 1/3 to ¼ of the iterative method

Summary

- Continuous-time Linear Quadratic Regulator (LQR) problem
- Kleinman's algorithm for the Algebraic Riccati Equation (ARE)
  - properties
- Discrete-time LQR problem
- Schur method for solving the ARE